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The Economics of Contests: Theory and Evidence

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A Bruno e Carlotta.

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Abstract of Thesis

This thesis consists of three chapters devoted to the study of the economics of contests. Each chapter can be read independently. A special attention is placed on teams' behaviour and team-incentive schemes. These questions are particularly important as the way in which institutions reward individuals shapes the inequality of the group to which these individuals belong.

Chapter 1. Optimal Prize Allocations in Group Contests.

We characterize the optimal prize allocation, namely the allocation that maximizes a group's effectiveness, in a model of contests. The model has the following features: (i) it allows for heterogeneity between and within groups; (ii) it classifies contests as “easy” and “hard” depending on whether the marginal costs are concave or convex. Thus, we show that in an “easy” contest the optimal prize allocation assigns the entire prize to one group member, the most skilled one. Conversely, all group members receive a positive share of the prize when the contest is “hard” and players have unbounded above marginal productivities. If the contest is “hard” and the marginal productivities are bounded above, then only the most skilled group members are certain of receiving a positive share of the prize for any distribution of abilities. Finally, we study the effects of a change in the distribution of abilities within a group. Our analysis shows that if the contest is either “easy” or a particular subset

of “hard”, then the more the heterogeneity within a group, the higher its probability of winning the prize.

Chapter 2. Inequalities within Groups: Theory and Evidence.

We study the design of a team in multi-team contests. Is it better to distribute prizes among players equally, or to just one player? And is it better to spend a budget on a diverse team with stars and rookies, or on an equal team? First, we study these questions theoretically. We find that depending on the production function, it is either optimal to (i) hire superstars and rookies, and reward superstars the most, or (ii) hire a homogeneous team and reward everyone equally. Then, we test the first set of predictions in the lab. Unlike the theory, superstars or concentrated rewards alone do not help a team win. Both must be used together.

Chapter 3. Model of War of Attrition with Outside Options.

We study a model of war of attrition with outside options. In a society that allocates rewards via tournaments, individuals decide how much resources dedicate towards winning the prize. Conflicts are of incomplete information and agents’ type consist of their drawn valuation of the prize and valuation of the outside option. We show that this model can be reduced to a standard war of attrition with one signal. Further, we derive the symmetric perfect Bayesian equilibrium of the game and discuss possible applications.

Lay Summary

In our society, a colossal amount of resources is allocated via contests. Examples include politicians competing in electoral campaigns, workers competing for job promotions, athletes for medals, countries for territories and natural resources. In all these examples, individuals, or group of individuals, expend costly effort and resources in order to win a prize.

In competition between groups, respective members win or lose the prize collectively. However, individuals in a group may have different skills. Therefore, we develop a theoretical model to investigate how to best split the prize among heterogeneous members in order to maximise the group's chance of winning. According to intuition, higher share of the prize should go to highly members of the groups. The findings of our model show exactly this. Furthermore, we find that the difference between the share of the prize of a highly skilled member and the share of a lower skilled one depends on the players' cost of effort. If a player's cost of effort increases quickly enough, then highly skilled players and lower skilled one split the prize almost equally.

With the support of a theoretical framework and a laboratory experiment, we also investigate whether it is better to spend a budget on a diverse team, with highly skilled members and lower skilled ones, or on an equal team. We discover that, depending on players' cost of contributing, it is either optimal to build a diverse team, and reward highly skilled players the most, or hire a homogeneous team and

reward everyone equally. We test the first set of predictions in the lab and find that assigning higher shares of the prize to highly skilled players substantially increases a group's performance. As a result, a diverse group has higher chances of winning than an equal group.

The last topic we study regards contests between individuals. Specifically, we develop a model to understand how a player's outside option affects her effort decision in a contest for a prize. Consider, for example, the investments made by two firms competing in to discover a new product. The volume of the investments depends on the profits the firms would make being the first to design the product, but also on the profits they would make if the patent is won by someone else. Our analysis reveals that players' contribution depends both on how much they value the prize and the outside option. As a result, the winner of the contest is not the player that values the prize the most, but the player with the highest difference between her value of the prize and the value of the outside option.

*“Un’idea, finché resta un’idea, è soltanto un’astrazione, se potessi mangiare un’idea
avrei fatto la mia rivoluzione.”*

- Giorgio Gaber

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Chapter 1

Optimal Prize Allocations in Group Contests

1.1 Introduction

“Soldiers generally win battles; generals get credit for them.”¹

Contests are ubiquitous in that they arise, for example, in wars, sports, electoral campaigns and workplace competitions. When contests arise between groups, respective members win or lose the prize collectively. However, individuals in a group may have different positions, skills, prize valuations, and various impacts over the outcome of the competition. When the prize has private characteristics, this within group heterogeneity can lead to personalized incentives, such as highly skilled players receiving a higher share of the prize over lower skilled ones. Conversely, an egalitarian allocation of prizes may be used to encourage cooperative behaviours among members.

At the beginning of the Republic of Rome, for example, the tribunes equally distributed the spoils of war among all army members, including those who only guarded the settlements and protected the wounded. After 407 BC, the Roman Senate intro-

¹Napoleon (1769-1821).

duced different incentives according to the roles performed by the members of the army: ordinary soldiers received a third of the wage of the knights and half of the wage of centurions. A similar rule was used under Napoleon’s Empire. Nowadays, victorious soldiers are awarded medals according to their rank in the army. Likewise, candidate prime ministers assign party members to different ministries. The assignment of the most influential people to the key ministries affects their efforts during the campaign and eventually the outcome of the elections. Finally, collective competitions are pervasive in the workplace, where, for example, department stores, retail chains, sales and production departments set up monetary rewards for the most productive teams. Nationally representative surveys reveal that 52% of firms use teamwork in the US, and 47% of British firms organized more than 90% of their workforce into teams; and 70% of Fortune 1000 companies use some form of team incentive (Bandiera et al., 2011).

In general, it seems evident that the way in which members are rewarded according to their roles and responsibilities affects how much they contribute to the group goal. Thus, the purpose of this article is twofold: to propose an allocative rule that maximizes group effectiveness, hereafter the “optimal prize allocation”; and to study how a change in the distribution of abilities within a group affects its effectiveness. We study our questions using a model with the following features: it allows for between and within groups heterogeneity; it classifies contests as “easy” and “hard” depending on whether the marginal costs are concave or convex; and each group has a manager that announces the optimal prize allocation only to her members.

Our analysis reveals the following results. In easy contests, the optimal prize allocations assigns the entire prize to one group member, the most skilled one. On the other hand, in hard contests in which players’ marginal productivities are unbounded above, it rewards all group members. If the contest is hard and players’ marginal

productivity are bounded above, then only the most skilled members of a group are certain of receiving a positive share of the prize for any distribution of abilities. Furthermore, we find that a change in the distribution of abilities within a group affects the probability of winning of all competing groups. Specifically, if the contest is either easy or a specific subset of hard, then the more the heterogeneity within a group, the higher the group probability of winning. Finally, we rank the probability of winning of the competing groups from highest to lowest under two prize divisions: the “egalitarian” allocation, i.e. the prize is equally shared among members of the same group; and the optimal prize allocation. Surprisingly, the ranking resulting from the egalitarian allocation can be fully reversed by implementing the optimal prize allocation.

Literature Review

Much progress has been made in the study of contests since the seminal work of Tullock (1980).²

In regards to group contests for public good, related set-ups are analyzed by Baik (1993, 2008) and Ryvkin (2011). The former shows that if players have linear cost, then only the most skilled member in every group contributes to the group cause. The latter, however, shows that all group members are active participants if costs are strictly convex. Moreover, Ryvkin (2011) studies how a contest organizer has to sort (heterogeneous) players in same size groups to maximize the aggregate effort exerted in the competition. His results are that if the players’ cost function is moderately (sufficiently) steep, then a more (less) balanced competition increases aggregate effort. Thus, one could wrongly assume that the same result extends to the optimal prize allocation because both papers relate to the steepness of the cost

²For a review see Corchón (2007).

function. However, this is not the case since the two definitions of steepness differ substantially.³ In addition, we do not focus on maximizing aggregate effort, for example by allowing groups to compete for different prizes, rather on how a group manager strategically chooses to split the prize among her members.

In regards to group contests for private good, the literature has considered the following ways of prize division among the winning group members: the “egalitarian” rule, used among others, by Esteban and Ray (2001) and Cheikbossian (2012) to study the group size paradox; the “relative effort” rule, which works as an incentive device, analysed by Nitzan (1991a); and any linear combination between the “egalitarian” and the “relative effort” rule studied by Nitzan (1991b) and Nitzan and Ueda (2011, 2018), meaning part of the prize is divided equally (egalitarian rule) and the rest proportionally according to each member’s effort (relative effort rule). Since the use of a relative effort allocation puts members of the same group in competition for the internal division of the prize, its full implementation eliminates the free-riding problem. However, its use effectively assumes that relative efforts can be costlessly observed and rewarded. Alternatively, a model with costs of monitoring needs to be introduced as in Ueda (2002). Even though this requirement seems innocuous, it reduces the applicability of this incentive device to few cases.⁴ Conversely, the egalitarian rule does not require that the individual contributions are observable, but it clearly tempts group members to free-ride on other’s contributions because they win or lose the prize as a group, i.e. winning the share of the prize is a “collective good”.

To the best of our knowledge, the few works that study allocative schemes and do not require monitoring, assume symmetry among players and focus on the effects

³For instance, all (convex) power functions are “moderately” steep in Ryvkin’s model. On the other hand, we define a contest as easy or hard depending on whether the marginal costs are concave or convex.

⁴See Bandiera et al. (2011) for an example about fruit picking.

of within group inequality. Nitzan and Ueda (2014) focus on the effects of intra-group heterogeneity in prize shares. The authors find that in easy (hard) contests the greater (lesser) the inequality in prize share, the higher the group efficiency. Cubel and Sanchez-Pages (2014) demonstrate through Atkinson's index of inequality that egalitarian groups have a higher probability of winning the contest when the efforts of the group members are complementary, or the contest is hard. There are two main differences between these models and our own: first, while they study exogenous variations in prize allocations, we analyse the endogenous choices of the optimal one; second, in our model players can be heterogeneous in their ability. If players are symmetric, group members choose the same amount of contribution in equilibrium. On the other hand, if they are heterogeneous, they react to the same incentive differently. As a matter of fact, in our model different levels of within group heterogeneity affects both the distribution of the prize among group members and the probability of winning of all groups.

Section 1.2 contains the preliminaries of the model; Section 1.3 presents our model of contests with managers and discusses the effect of within group heterogeneity; Section 1.4 concludes.

1.2 Model preliminaries

We first analyse a model of *complete* information and *exogenous* prize allocations so as to state an equilibrium existence result useful for Section 1.3, where we will introduce *incomplete* information and *endogenous* prize allocations. This approach is convenient both to introduce the preliminaries of our game with managers and to compare efforts under the implementation of the egalitarian rule and the optimal one. Thus, we consider a game with N groups. The i -th group is formed by n_i risk-neutral

individuals making a total of $\sum_i^N n_i$ players. Players within-groups are indexed by $ik = (i1, \dots, in_i)$. All players simultaneously and irreversibly exert an effort $x_{ik} \geq 0$. The group effort is the linear sum of its members' effort, $X_i = \sum_{k=1}^{n_i} x_{ik}$. The group probability of winning is defined by the Tullock success function $\sigma_i = X_i/X$, where $X = \sum_{i=1}^N X_i$. Exerting effort is costly, but individuals are (possibly) heterogeneous in their abilities, $v_{ik} \in (0, \infty)$. The cost of effort is given by $v_{ik}^{-1}g(x_{ik})$, and thus it is costlier for low ability individuals to exert effort.⁵ We impose the following assumption on $g(x)$:

Assumption 1 i) $g(0) = 0$; ii) $g'(0) = 0$; iii) $g'(x) > 0$ for all $x > 0$; iv) $g''(x) > 0$ for all $x > 0$; v) $g'''(x)$ exists for all $x > 0$.

Part (i) states that players do not bear costs when they do not exert any effort. Part (ii) states that the marginal cost of effort at $x = 0$ is zero. Part (iii) and (iv) state respectively that the effort cost function is strictly increasing and strictly convex, which ensures the existence and uniqueness of an equilibrium in which all players exert a positive effort, as long as they receive a strictly positive prize. Finally, part (v) is necessary for comparative statics. Moreover, since g' is monotonic and continuous, it has a well-defined inverse function, $f = (g')^{-1}$. Assumption 1 is held throughout the paper.

In our setting, the winning group is rewarded with a private good prize normalized to one, and the losing groups receive zero. For the moment, we also assume that the winning ik member receives a share of the prize ϕ_{ik} according to an *exogenous* prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ s.t. $\sum_{k=1}^{n_i} \phi_{ik} = 1$. In light of this, the player ik 's expected payoff is

⁵This approach to define heterogeneity is commonly used in the literature of contests, see for example Ryvkin (2011, 2013), Brookins et al. (2015) and Nitzan and Ueda (2018).

$$\pi_{ik} = \frac{X_i}{X} \phi_{ik} - \frac{g(x_{ik})}{v_{ik}}. \quad (1.1)$$

Each player ik 's best response to all other players' choice of effort is given by the first-order condition associated with the maximization of π_{ik} as a function of x_{ik} , subject to $x_{ik} \geq 0$. Since (1.1) is strictly concave with respect to x_{ik} , the first-order condition is necessary and sufficient for the best response. It follows that the player ik 's best response is

$$\frac{X_{j \neq i}}{X^2} \phi_{ik} v_{ik} = g'(x_{ik}). \quad (1.2)$$

As discussed in the introduction, a contest for public good and linear costs, $g(x_{ik}) = x_{ik}$, is considered in Baik (2008). The result is that in each group only the player with the lowest marginal cost exerts a positive effort.⁶ On the other hand, under Assumption 1, it is possible to show that there is a pure strategy Nash equilibrium in which all players that receive a positive share of the prize are active participants in equilibrium.

Lemma 1.1. *Under Assumption 1, the contest between groups has a unique Nash equilibrium in pure strategies for any prize allocation. In equilibrium, at least one player in each group exerts a positive effort, therefore all groups exert a positive effort. The equilibrium effort x_{ik}^* satisfies the system of Equation (1.2) with equality, and defines the group i 's effort as*

$$X_i^*(\phi_i) = \sum_{k=1}^{n_i} x_{ik}^* = \sum_{k=1}^{n_i} f \left(\frac{X_j^*(\phi_j)}{(X^*)^2} \phi_{ik} v_{ik} \right). \quad (1.3)$$

⁶A model with linear costs can be considered as a special case of the easy contests presented in this paper. As a result of the stark free-riding, it would be optimal to allocate the entire prize to the player with the lowest marginal cost.

Finally, we define contests in the following two ways: we refer to a contest as “easy” when the cost of exerting an additional amount of effort does not rapidly increase; conversely, a contest is “hard” when an additional amount of effort leads to a significant increase in the marginal cost.⁷ Formally,

Definition 1. A contest is “easy” when g' is strictly concave (f convex). Conversely, a contest is “hard” when g' is strictly convex (f concave).

1.3 A model of group contests with managers

Hereafter, we move away from the model with complete information and exogenous predetermined prize allocations. Instead, we now assume that every group has a manager that sets a prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ s.t. $\sum_{k=1}^{n_i} \phi_{ik} = 1$ in order to maximize her group’s probability of winning.⁸ This is a common situation since the compensation of managers is usually aligned with the results of their group. In addition, for a matter of realism, we assume that the prize allocation implemented by each manager is *unobservable* by those belonging to other groups. However, all players’ abilities remain common knowledge.

Information Structure

The timing and the information structure are adapted from Nitzan and Ueda (2011, 2018) and described as follows: i) each manager announces the prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ to her members and ii) group members enter in the contest without knowing the prize allocations implemented in the other groups, and determine their

⁷The form of the marginal cost, g' , depends on its third derivative. There are contests that are neither “easy” nor “hard”, for example when $g'''(x) > 0$ for some x , and $g'''(x) < 0$ for others. However, we focus our analysis only on these two cases.

⁸We could relax the equality constraint to $\sum_{k=1}^{n_i} \phi_{ik} \leq 1$, but it would not change our results since we focus on the case in which managers aim to maximize their group’s effort.

contributions simultaneously and noncooperatively. Before choosing his contribution, group member ik only knows his own group prize allocation, and finds himself in the information set containing the nodes at which the other groups have chosen a prize allocation $\phi_{j \neq i} = (\phi_{j1}, \dots, \phi_{jn_j})$. Thus, member ik 's strategy is described as a function of ϕ_i , and denoted by $x_{ik}(\phi_i)$.

Beliefs

We consider the perfect Bayesian equilibrium as a solution of our model assuming that players can use only pure strategies. Since the choice of the prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ is simultaneously made at the beginning of the game by the managers, then group members' beliefs are trivial. The belief of player ik , denoted by $\mu_{ik}(\phi_i)$, is a probability distribution defined over the space of possible allocations implemented in other groups. Suppose that $(\phi_1^*, \dots, \phi_N^*)$ is an equilibrium prize allocation. At the information set lying on the equilibrium path the requirement of consistency implies that player's ik belief satisfies $\mu_{ik}(\phi_{j \neq i}^* | \phi_i^*) = 1$. Finally, we restrict the beliefs of group members off the path appealing to the "no-signalling-what-you-don't-know" condition.⁹ Thus, any deviation by a manager does not change the beliefs of her group members about the allocations implemented in other groups, i.e. $\mu_{ik}(\phi_{j \neq i}^* | \phi_i) = 1 \forall \phi_i$. Altogether, we can use Equation (1.2) to characterize player ik and group i 's best responses. Since members are aware of the prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ implemented by their own manager, i.e. at the information set indexed by ϕ_i , then the best responses are

$$X_i^*(\phi_i) = \sum_{k=1}^{n_i} x_{ik}^*(\phi_i) = \sum_{k=1}^{n_i} f \left(\frac{X_{j \neq i}^*(\phi_j^*)}{X^2} \phi_{ik} v_{ik} \right). \quad (1.4)$$

As previously established, players ik 's expected payoff is strictly concave in x_{ik} . It

⁹Fudenberg and Tirole (1991).

follows that the first-order conditions given by Equation (1.4) are necessary and sufficient for the best responses.

1.3.1 Efficient managers

Managers' behaviours vary accordingly to their objectives. We consider the case in which they want to maximize their group's effort through the prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$. Before moving forward, it is important to note that the members of a group hold identical beliefs about the prize allocations implemented in other groups. Thus, if a manager maximizes her own group effort $X_i(\phi_i)$, then she also maximizes her group probability of winning since $\sigma_i = X_i(\phi_i)/(X_i(\phi_i) + X_j^*(\phi_j^*))$. This observation allows us to see that the maximization of the group effort $X_i(\phi_i)$ and the maximization of the group probability of winning σ_i are two equivalent problems. Overall, the manager of group i has to solve

$$\phi_i^* \in \operatorname{argmax} X_i^*(\phi_i) \text{ s.t. } \sum_{k=1}^{n_i} \phi_{ik} = 1, \phi_{ik} \geq 0 \forall k, \quad (1.5)$$

where $X_i^*(\phi_i)$ is defined by Equation (1.4). Hence, if we find a profile of prize allocations $(\phi_i^*, \dots, \phi_N^*)$ that solves (1.5) for all i , and all players maximize their expected payoff under their information set, i.e. Equation (1.4) holds with equality for all i , then we can state that it is a pure-strategy perfect Bayesian equilibrium of the model with managers. As established by the following propositions, the number of equilibria depends on the type of contest and group members' ability.

Proposition 1.3.1. *Given Assumption 1, the easy contest between groups with managers has $\prod_i^N n_i^h$ perfect Bayesian equilibria in pure strategies, where n_i^h is the number of group members with the highest ability in group i . Moreover,*

- i) every ϕ_i^* rewards the entire prize to one of the n_i^h group members;*

ii) all equilibria provide the same X_i^* , σ_i^* , and $X^* \forall i$.

Since an equilibrium is a profile of prize allocations $(\phi_i^*, \dots, \phi_N^*)$ that solves the systems of equations (1.4) and (1.5), and ϕ_i^* rewards the entire prize to one among the most able group members, then all the optimal allocations provide the same X_i^* for all i . However, at different equilibrium allocations different members contribute to the group effort. On the other hand, a unique equilibrium with one optimal prize allocation exists if the contest is hard.

Proposition 1.3.2. *Given Assumption 1, the hard contest between groups with managers has a unique perfect Bayesian equilibrium in pure strategies. Moreover,*

- i) *symmetric players are rewarded equally, $v_{ik} = v_{im}$ implies $\phi_{ik}^* = \phi_{im}^*$;*
- ii) *if marginal productivity is unbounded, then all members receive a positive share of the prize, $g''(0) = 0$ implies $\phi_{ik}^* > 0 \forall k$;*
- iii) *if marginal productivity is bounded above, then only the highest ability members receive a positive share of the prize for any equilibrium effort, $g''(0) > 0$ implies $\phi_{ik}^* > 0 \forall X^*$ iff $v_{ik} = \max[v]$.*

In hard contests, a prize allocation is optimal when all the members of a group have the same marginal productivity, i.e. if $\frac{v_{ik}}{g''(x_{ik}^*)} = \frac{v_{im}}{g''(x_{im}^*)}$ where $m \neq k$.¹⁰ Thus, group members with the same ability receive the same share of the prize ϕ_{ik}^* . Furthermore, if $g''(0) = 0$, then all players always receive a positive share of the prize since their marginal productivity at zero effort equals infinity. Conversely, if $g''(0) > 0$, then the players' marginal productivity are bounded above. As a result, only the highest

¹⁰Equivalently, for any two active members it holds $\frac{g'(x_{i1}^*)}{g''(x_{i1}^*)\phi_{i1}^*} = \frac{g'(x_{im}^*)}{g''(x_{im}^*)\phi_{im}^*} \forall m > 1$.

ability members of every group are certain of receiving a positive share of the prize. For example, we may have that $\frac{v_{i1}}{g''(X_i^*)} \geq \frac{v_{im}}{g''(0)}$, i.e. player $i1$ receives the entire prize because we do not allow for negative prizes. We can relate our results to Cornes and Hartley (2005). In general terms, the authors define a player's *dropout point* as the equilibrium effort X^* that makes him a non-active participant. If $g'(0) = 0$, then there is no dropout point and all players exert positive effort. As a matter of facts, in our setting, all groups are always active, but member im drops out if $v_{im} < v_{i1}$ in easy contests, and if $v_{im}/g''(0) \leq v_{i1}/g''(x_{i1}^*) \forall m > 1$ in hard ones. However, group members do not drop out from the competition voluntarily, but they optimally respond to the allocation implemented by their manager. Part (iii) of the proposition can be related to the group size effect in contests. Specifically, the literature shows that, in hard contests in which groups are formed by symmetric members who equally share the prize, a group increases its effort increasing its size.¹¹ In contrast, in our setting that introduces heterogeneity, an increase in size affects the group effort if and only if the new member receives a positive share of the prize from the optimal prize allocation. Finally, the two propositions above relates to inequality if we interpret the optimal allocation as the efficient (reverse) Pigou-Dalton transfer of the (possible) rewards.¹² Thus, when the contest is easy and/or players are heterogeneous, then managers have always a preference for inequality. Specifically, they would commit to transfer part of (or all) the possible gains from the less able workers to the most able ones even if the contest is hard.

¹¹See the “anti-Olson (2009) theorem” in Nitzan and Ueda (2014) and Proposition 2 in Esteban and Ray (2001).

¹²I am grateful to an anonymous referee for bringing this point to my attention.

Within group heterogeneity and group effectiveness

In this section, we revise some results of the effects of within group heterogeneity in ability under the widely used egalitarian allocation,¹³ $\phi_{ik} = 1/n_i \forall k$, and then compare them to the effects of heterogeneity under the optimal allocation, $\phi_i = \phi_i^* \forall i$. To consider this matter, we define heterogeneity within groups following the definition of inequality in the sense of Lorenz dominance. Thus, we ask when $X'_i(x'_{ik}, \dots, x'_{in_i}) \geq X_i(x_{i1}, \dots, x_{in_i})$, where $x'_{i1}, \dots, x'_{in_i}$ are members equilibrium efforts under a “more spread out” distribution of abilities than x_{i1}, \dots, x_{in_i} . To go along with this analysis we define the notions associated with majorization introduced by Hardy et al. (1934) which is equivalent to the notion of inequality in the sense of Lorenz dominance as shown by Dasgupta et al. (1973).

Definition 2. Let \mathbf{x}' and \mathbf{x} be two vectors in R^n , ordered so that $x'_1 \geq \dots \geq x'_n$ and $x_1 \geq \dots \geq x_n$. If $\sum_{k=1}^n x'_k = \sum_{k=1}^n x_k$ and $x'_1 + \dots + x'_l \geq x_1 + \dots + x_l$ for all $l \leq n$ (with strict inequality for at least one l), then we say that \mathbf{x}' majorizes \mathbf{x} written as $\mathbf{x}' \succ \mathbf{x}$. A permutation symmetric function F of n variables is Schur-convex if the inequality $F(\mathbf{x}') \geq F(\mathbf{x})$ holds whenever $\mathbf{x}' \succ \mathbf{x}$. General discussion of majorization theory and Schur-convex functions can be found in Marshall et al. (1979).

Thus, we try to understand in which situations within group heterogeneity increases group effectiveness drawing on existing knowledge from non-strategic environments, but keeping in mind that we actually move towards different equilibria. Let us now assume that all managers implement the egalitarian allocation, $\phi_i = 1/n_i \forall i$. In this situation, the group i 's effort in equilibrium is

¹³This is mathematically equivalent to a public good contest.

$$X_i^* = \sum_{k=1}^{n_i} f\left(\frac{1 - \sigma_i^*}{X^*} \frac{1}{n_i} v_{ik}\right). \quad (1.6)$$

Fixing σ_i^* and X^* , the group i 's effort can be written as a function of the vector of abilities, $X_i^* = F(\mathbf{v}_i)$. This observation together with Definition 2 helps us to state¹⁴

Lemma 1.2. *Given a contest between groups in which the prize is equally shared among group members:*

- i) if the contest is easy, then the higher the within group heterogeneity in ability, the higher the group effectiveness. Formally, a change from \mathbf{v}_i to \mathbf{v}'_i where $\mathbf{v}'_i \succ \mathbf{v}_i$ implies $\sigma'_i > \sigma_i$, $X' > X$ and $X'_i > X_i$.*
- ii) if the contest is hard, then the lower the heterogeneity in ability, the higher the group effectiveness. Formally, a change from \mathbf{v}_i to \mathbf{v}'_i where $\mathbf{v}'_i \prec \mathbf{v}_i$ implies $\sigma'_i > \sigma_i$, $X' > X$ and $X'_i > X_i$*

The intuition of this result is easy to grasp: when the contest is easy, low ability players free-ride on high ability ones who have a lower cost of contributing. On the other hand, high ability players are willing to exert substantial amounts of effort that more than compensate for the free-riding since their cost (for additional contributions) does not increase rapidly. Hence, keeping the average group ability constant, the greater the heterogeneity within a group, the higher its effectiveness, or, equivalently, a more spread out (unequal) distribution of abilities within a group increases its effectiveness. In hard contests, however, the cost of additional amounts of effort increases so rapidly that works as a deterrent for all players, but especially for the highly skilled that recede from exerting substantial contributions. So, when

¹⁴Lemma 2 can be derived from Proposition 2 in Nitzan and Ueda (2014) where they allow heterogeneity in ϕ_{ik} fixing v_{ik} .

players are equally rewarded, a lower heterogeneity weakens this effect by making players exert similar (and less costly at the margin) efforts. Similarly, a less spread out distribution of abilities implies that members share the costs more equally, thus paying a lower cost per unit of group effort.

A natural question to ask is whether the above result extends to our framework with optimal incentives where managers, assigning specific incentives, can enlarge the range of the possible contributions of their members, which now depends both on abilities and prize shares. The study of this matter under the implementation of the optimal allocation ϕ_i^* is straightforward for easy contests because the highest ability players always receive the entire prize. On the other hand, for hard contests, we have to carefully analyse players' and managers' behaviours. Indeed, any change in ability distribution leads to a change in the optimal allocation ϕ_i^* , together with changes in the group effort X_i^* and aggregate effort X^* . Thus, to make this analysis tractable we assume that $g''(0) = 0$, which implies that all group members always receive a positive share of the prize, i.e. $\phi_{ik}^* > 0 \forall v_{ik} > 0$.¹⁵ Altogether, the equilibrium group i 's effort under the optimal prize allocation is

$$X_i^* = \sum_{k=1}^{n_i} f\left(\frac{1 - \sigma_i^*}{X^*} v_{ik} \phi_{ik}^*\right). \quad (1.7)$$

Since the prize allocation is a function of the distribution of abilities within the group, fixing σ_i^* and X^* allows us to write the group i 's effort as a function of the vector of abilities, $X_i^* = F(\mathbf{v}_i)$. This observation and Definition 2 help us to state the next proposition.¹⁶

¹⁵The widely used power cost function x^α/b satisfies this condition.

¹⁶Condition on part (ii-iii) is similar to measures of cautiousness, i.e. $(g'''/g'')/(g''/g')$. Similar versions of it appears prominently in the contest literature, for example see lemma 3 in Akerlof and Holden (2012) and Proposition 1 in Ryvkin (2011). However, $g(x)$ is a deterministic cost function, hence these notions are not directly relevant to the situation under study.

Proposition 1.3.3. *Given a contest between groups with managers:*

- i) if the contest is easy, then an increase in within group heterogeneity raises the group effectiveness. Formally, a change from \mathbf{v}_i to \mathbf{v}'_i where $\mathbf{v}'_i \succ \mathbf{v}_i$ implies $\sigma'_i \geq \sigma_i$, $X' \geq X$, and $X'_i \geq X_i$;*
- ii) if the contest is hard, $g''(0) = 0$ and $g''(x)/g'(x) > 2g'''(x)/g''(x)$, an increase in within group heterogeneity raises the group effectiveness. Formally, a change in the distribution of ability from \mathbf{v}_i to \mathbf{v}'_i where $\mathbf{v}'_i \succ \mathbf{v}_i$ implies $\sigma'_i > \sigma_i$, $X' > X$, and $X'_i > X_i$;*
- iii) if the contest is hard, $g''(0) = 0$ and $g''(x)/g'(x) < 2g'''(x)/g''(x)$, then a decrease in within group heterogeneity raises the group effectiveness. Formally, a change in the distribution of ability from \mathbf{v}_i to \mathbf{v}'_i where $\mathbf{v}'_i \prec \mathbf{v}_i$ implies $\sigma'_i > \sigma_i$, $X' > X$, and $X'_i > X_i$;*

Common to the literature of contests is the assumption that either groups implement the egalitarian allocation or group members are symmetric in their ability. Thus, the analysis of within group heterogeneity follows directly from Lemma 1.2. As a matter of fact, it is widely argued in the literature that in hard contests groups are more effective the less their within group heterogeneity. Examples include Nitzan and Ueda (2014), Cubel and Sanchez-Pages (2014), and Esteban and Ray (2001). However, as shown by part (ii), these results do not hold under the implementation of the optimal rule. For example, let the cost function be x^α , and the related marginal costs $g'(x) = \alpha x^{(\alpha-1)}$. It is easy to see that a contest is hard for any $\alpha > 2$, and $g''(x)/g'(x) > 2g'''(x)/g''(x) \forall \alpha \in (2, 3)$, which implies that for $2 < \alpha < 3$ within group heterogeneity increases group efficiency under the use of the optimal allocation. Thus, the deterrent effect that induced highly skilled players to recede

from exerting substantial contributions can be balanced out by assigning them higher shares of the prize.¹⁷ Indeed, if the initial share is the same, then an increment of the allocated prize induces more effort from the more able individual. Altogether, a more spread out distribution of abilities is efficient if the complementarity between ability and rewards boosts enough skilled players efforts to more than compensate for the deterrent effect of the increase in costs.¹⁸ We conclude our analysis highlighting other relevant results related to the use of the optimal prize allocation. As shown in the following propositions, which follow directly from Proposition 1.3.3, we can rank groups' probability of winning. Such a ranking is not possible in easy contests with heterogeneous groups under the implementation of other incentive mechanisms such as the relative effort rule and the egalitarian rule.

Proposition 1.3.4. *Consider a contest with N groups formed by n_i individuals such that $\mathbf{v}_1 \succ \dots \succ \mathbf{v}_N$. If the contest is easy and $\phi_i = \phi_i^* \forall i$, then groups' probability of winning can be ordered according to the highest ability members in every group. Formally, $v_{i1} > \dots > v_{N1}$ implies $\sigma_1 > \dots > \sigma_N$;*

Thus, if managers optimally allocate the prize, differences in sizes between groups are irrelevant to the group efficiency. Indeed, groups can be ranked according to the most skilled member in every group. In addition to this, for some specific hard contests, the ranking resulting from the egalitarian allocation can be fully reversed by implementing the optimal prize allocation.

¹⁷If $g''(x)/g'(x) > 2g'''(x)/g''(x)$, then $g'(x)/g''(x)$ is increasing (see equation (1.23)), which implies that higher ability players receive higher share of the prize. Interestingly, this is not generally true.

¹⁸The same intuition can be explained looking at the marginal costs of contributing per unit of group effort via a simple example: a highly skilled player with ability V and a group of n symmetric players with ability V/n are competing for a prize P . The marginal cost of *group effort* are $\frac{X^{\alpha-1}}{V}$ and $(\frac{X}{n})^{\alpha-1} \frac{n^2}{V}$ respectively. Finally, it is easy to see that if $1 < \alpha < 3$ the single player has lower costs and then he exerts higher effort in equilibrium.

Proposition 1.3.5. *Consider a hard contest with N groups formed by $n_i = n$ individuals such that $\mathbf{v}_1 \succ \dots \succ \mathbf{v}_N$. Let $g''(0) = 0$, and $g''(x)/g'(x) > 2g'''(x)/g''(x)$, then the use of the egalitarian allocation $\phi_i = 1/n$ implies $\sigma_1 < \dots < \sigma_N$, while the optimal allocation $\phi_i = \phi_i^*$ implies $\sigma_1 > \dots > \sigma_N$.*

A full example with power cost functions

Let $g(x) = x^\alpha$, which for $\alpha > 1$ satisfies Assumption 1. Group i 's best response, when members are aware of the prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ implemented by their own manager, is given by

$$X_i^*(\phi_i) = \sum_{k=1}^{n_i} x_{ik}^*(\phi_i) = \sum_{k=1}^{n_i} \left(\frac{X_{j \neq i}^*(\phi_j^*)}{X^2} \phi_{ik} v_{ik} \right)^{\frac{1}{\alpha-1}}. \quad (1.8)$$

In order to maximize their group's effort managers have to solve

$$\phi_i^* \in \operatorname{argmax} X_i^*(\phi_i) \text{ s.t. } \sum_{k=1}^{n_i} \phi_{ik} = 1, \phi_{ik} \geq 0 \forall k. \quad (1.9)$$

From Proposition 1.3.2 we know that $\phi_{ik}^* > 0 \forall k$ if $\alpha \in (2, \infty)$ (hard contest), while $\phi_{i1} = 1$ if $\alpha \in (1, 2)$ (easy contest). Moreover, having assumed a specific cost function, the solution of (1.9) is

$$\phi_{ik}^* = \frac{v_{ik}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n_i} v_{ik}^{\frac{1}{\alpha-2}}}. \quad (1.10)$$

Finally, substituting (1.10) into (1.8) and rearranging, gives us the group i efforts in equilibrium of our model of contests with managers.

$$X_i = \begin{cases} \left(\frac{1-\sigma_i^*}{\alpha X^*} v_{i1} \right)^{\frac{1}{\alpha-1}} & \text{if } \alpha \in (1, 2] \\ \left(\frac{1-\sigma_i^*}{\alpha X^*} \right)^{\frac{1}{\alpha-1}} \left(\sum_{k=1}^{n_i} v_{ik}^{\frac{1}{\alpha-2}} \right)^{\frac{\alpha-2}{\alpha-1}} & \text{if } \alpha \in (2, \infty) \end{cases} \quad (1.11)$$

Proposition 1.3.6. *Given a contest between groups in which the prize is optimally allocated among group members:*

- i) if $1 < \alpha < 3$, then the higher the heterogeneity in ability the higher the group effort.*
- ii) if $\alpha > 3$, then the lower the heterogeneity in ability the higher the group effort.*
- iii) If $\alpha \rightarrow \infty$, then $\phi_{ik} \rightarrow 1/n_i \forall i, k$.*

The above proposition shows that the category of contests for which heterogeneity increases group effectiveness is larger under the implementation of the optimal prize allocation than the egalitarian one. Specifically, it moves from $1 < \alpha < 2$, for the egalitarian allocation, to $1 < \alpha < 3$, for the optimal one. Finally, we can establish from Equation (1.10) that the share of the prize that players receive depends on parameter α as follows: the higher the α the more equal the prize division among group members. It follows that, when the contest gets extremely hard, the optimal allocation tends to the egalitarian rule.

1.4 Conclusions

We have examined a model of group contests for a private good, in which individual contributions are not observable, to provide a prize allocation that maximizes groups' effectiveness. Our main findings are the following: in easy contests it is optimal to allocate the entire prize to one of the most able group members; in hard contests the optimal allocation depends on players' ability and their marginal productivity of effort; we provide sufficient conditions that make, in contrast with other results in the literature, heterogeneous groups more effective than homogeneous groups even in contests with strictly convex marginal costs.

Our model is general in the sense that it can be applied to many types of conflicts and work environments which encourage competition through specific incentive schemes. Moreover, it does not require that managers observe the contributions of every group member, a requirement that is necessary to implement the relative effort rule. Thus, we can advise managers on how to assign incentives and build their teams in different situations. For instance, in competitions with symbolic rewards, such as “best store of the month”, we can advise managers to form a heterogeneous group to prevent free-riding problems when the cost function is not too steep. On the other hand, in retail firms that set up monetary reward contests for sales departments during periods with a positive shock or a peak in the demand for goods, such as the run-up to Christmas, we can suggest to the team managers to divide the prize among all group members to increase the team productivity assuming that the extra work provided by the workers substantially increases their marginal costs.

In addition, our analysis on within group heterogeneity reveals new insights on inequality in conflicts. Specifically, it shows that a more spread out distribution of ability increases group effectiveness for hard contests under the condition that groups implement the optimal prize allocation.

The simplicity of our framework is attractive but might be criticized because the group managers do not exert effort and have one goal: to maximize their own team’s effectiveness. We could have assumed that team managers may contribute to their group’s effort, but it would not have changed our intuitions based on the contest’s categorization. Moreover, it is implicit in the model that managers maximize group efficiency because this gives them some direct or indirect benefits aligned with their teams’ results; for example, job promotions or other monetary awards.

Appendix A

Proof of Lemma 1.1

The following proof is an extension of Ryvkin (2011) for a perfectly divisible prize. Player ik 's best response function has to satisfy

$$\frac{X_{j \neq i}}{X^2} = (v_{ik}\phi_{ik})^{-1}g'(x_{ik}). \quad (1.12)$$

Note that the left-hand side of Equation (1.12) is the same for any player k of group i . Without loss of generality, let $\phi_{i1} > 0$. It follows that for any x_{i1} we have $(v_{im}\phi_{im})^{-1}g'(x_{im}) = (v_{i1}\phi_{i1})^{-1}g'(x_{i1})$. Thus, the effort exerted by all im , where $m > 1$, can be uniquely determined as a share of the effort exerted by player $i1$ as

$$x_{im} = g'^{(-1)}\left(\frac{v_{im}\phi_{im}}{v_{i1}\phi_{i1}}g'(x_{i1})\right). \quad (1.13)$$

The group i 's effort X_i can be written as

$$\alpha_i(x_{i1}) = x_{i1} + \sum_{m>1}^{n_i} g'^{(-1)}\left(\frac{v_{im}\phi_{im}}{v_{i1}\phi_{i1}}g'(x_{i1})\right),$$

and using (1.13), the related marginal cost as

$$G'_i(x_{i1}) = \sum_k^{n_i} \frac{g'(x_{ik})}{v_{ik}}.$$

Functions $\alpha_i(x_{i1})$ and $G'_i(x_{i1})$ are strictly increasing and satisfies $\alpha_i(0) = 0$ and $G'_i(0) = 0$. Therefore, the contests among groups reduces to a contest among N individuals:

$$\frac{1 - \sigma_i}{\sum_{i=1}^N \alpha_i(x_{i1})} = G'(x_{i1}). \quad (1.14)$$

Let $X_i = \alpha_i(x_{i1})$, $x_{i1} = \alpha^{-1}(X_i)$. Define $G_i(X_i) = \int_0^{X_i} G'(\alpha^{-1}(t))dt$ with initial conditions $G_i(X_i) = 0$. $G_i(X_i)$ is strictly increasing, strictly convex and satisfies Assumption 1. The group i 's expected payoff can be written as

$$\frac{X_i}{X} - G_i(X_i). \quad (1.15)$$

The uniqueness of equilibrium follows from Theorem 3 of Cornes and Hartley (2005).

Proof of Proposition 1.3.1

Let $g'(x)$ be strictly concave and $f(x)$ strictly convex (easy contest). A profile of allocations $(\phi_1^*, \dots, \phi_N^*)$ is a perfect Bayesian equilibrium if it satisfies Equation (1.4) and (1.5) $\forall i$.

In order to prove the proposition we use the following observations:

- i) Equation (1.4) holds with equality for any prize allocation, i.e. for any $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ there exists only one X_i (the opposite is not true);
- ii) Equation (1.4) is a sum of strictly convex functions for any fixed $X = X_i + X_{j \neq i}^*(\phi_j^*) > 0$.¹⁹ Hence, it is strictly convex.

Let us define the group i 's effort that satisfies Equation (1.4) for a prize allocation ϕ_i' as $X_i(\phi_i', X_j^*(\phi_j^*)) = X_i'$ (see observation i). Then, we can find an alternative allocation, ϕ_i^a , which provides higher effort than ϕ_i' , as follows: we fix the total effort at $X = X_i' + X_j^*(\phi_j^*)$; and we maximize the group i 's effort given by Equation (1.4). Clearly, the solution of this maximization problem lies in a corner (see observation

¹⁹Assumption 1 guarantees that if $\phi_{ik} > 0$ for at least a k in every group, then $X_i > 0 \forall i$.

ii). There are n_i^h allocations, where n_i^h is the number of players with the highest ability in group i . Under the allocation ϕ_i^a we have that

$$X'_i < \sum_{k=1}^{n_i} f\left(\frac{X_j^*(\phi_j^*)}{(X'_i + X_j^*(\phi_j^*))^2} v_{ik} \phi_{ik}^a\right).$$

The first order condition is satisfied for $X_i^a > X'_i$ such that $X_i^a = \sum_{k=1}^{n_i} f\left(\frac{X_j^*(\phi_j^*)}{(X_i^a + X_j^*(\phi_j^*))^2} v_{ik} \phi_{ik}^a\right)$. It is straightforward that every manager maximizes her group effort iff $\phi_i = \phi_i^*$ for any $X_j^*(\phi_j^*) > 0$, i.e. she allocates the entire prize to one of the most able members. The uniqueness of equilibrium follows from Lemma 1.1.

Proof of Proposition 1.3.2

Let $g'(x)$ be strictly convex and $f(x)$ strictly concave (hard contest).

In order to prove the proposition we use the observation (i) and (ii) stated for the proof of Proposition 1.3.1. First, we define the group i 's effort that satisfies Equation (1.4) for a particular prize allocation ϕ'_i as $X_i(\phi'_i, X_j^*(\phi_j^*)) = X'_i$ using observation (i). Second, using observation (ii), we find an alternative allocation, ϕ_i^a , which provides higher effort than ϕ'_i as follows: we fix the total effort at $X = X'_i + X_j^*(\phi_j^*)$; and we maximize the group i 's effort given by Equation (1.4). The solution of this maximization problem is interior, unique and implies

$$X'_i < \sum_{k=1}^{n_i} f\left(\frac{X_j^*(\phi_j^*)}{(X'_i + X_j^*(\phi_j^*))^2} v_{ik} \phi_{ik}^a\right).$$

The first order condition is satisfied for $X_i^a > X'_i$ such that $X_i^a = \sum_{k=1}^{n_i} f\left(\frac{X_j^*(\phi_j^*)}{(X_i^a + X_j^*(\phi_j^*))^2} v_{ik} \phi_{ik}^a\right)$. Overall, we have to find the allocation that maximizes (1.5) at $X = X_i(\phi_i^*) + X_j^*(\phi_j^*)$, and the $X_i(\phi_i^*)$ that satisfies Equation (1.4). The solution of Equation (1.5) is given by

$$\frac{\frac{1-\sigma_i}{X}v_{ik}}{g''(x_{ik})} - \lambda = 0 \quad \forall k. \quad (1.16)$$

Let, without loss of generality, $v_{i1} \geq \dots \geq v_{in_i}$. From the system of equations (1.16) we have that

$$\frac{v_{i1}}{g''(x_{i1})} \geq \frac{v_{im}}{g''(x_{im})} \quad \forall m > 1. \quad (1.17)$$

Using (1.17), the group i 's effort $X_i(\phi_i^*)$ can be written as

$$\nu_i(x_{i1}) = x_{i1} + \sum_{m>1}^{n_i} \max \left[0, g''^{-1} \left(\frac{v_{im}}{v_{i1}} g''(x_{i1}) \right) \right], \quad (1.18)$$

and the related marginal costs as

$$G'_i(x_{i1}) = \sum_{ik}^{n_i} \frac{g'(x_{ik})}{v_{ik}}.$$

Functions $\nu(x_{i1})$ and $G'(x_{i1})$ are strictly increasing and satisfy $\nu(0) = 0$, $G'(0) = 0$.

Therefore, the group contests reduces to a contests among N individuals:

$$\frac{1 - \sigma_i}{\sum_{i=1}^N \nu_i(x_{i1})} = G'_i(x_{i1}). \quad (1.19)$$

Finally, let $X_i = \nu_i(x_{i1})$, $x_{i1} = \nu_i^{-1}(X_i)$. Define $G_i(X_i) = \int_0^{X_i} G'(\nu_i^{-1}(t))dt$ with initial conditions $G_i(X_i) = 0$. $G_i(X_i)$ is strictly increasing, strictly convex and satisfies Assumption 1. Group i 's expected payoff can be written as

$$\frac{X_i}{X} - G_i(X_i). \quad (1.20)$$

The uniqueness of equilibrium follows from Theorem 3 of Cornes and Hartley (2005).

The equilibrium effort X_i^* implicitly defines the equilibrium allocation $(\phi_1^*, \dots, \phi_{N_i}^*)$ through the relation $X_i^* = \nu_i(x_{i1}^*)$. To better see that this solution gives a unique prize

allocation for all i consider the following contradiction. Suppose that the equilibrium effort X_i^* can be obtained by two prize allocations, ϕ_i^1 and ϕ_i^2 . This would imply that setting $X^* = X_i^* + X_j^*(\phi_j^*)$, and solving for Equation (1.5) gives the two solutions ϕ_i^1 and ϕ_i^2 . However, due to the strictly concavity of the best response function for fixed X the solution is interior and unique.

Part i)

Recall that $X_i^*(\phi_i) > 0$. Then, it is straightforward to see that the system of equations (1.17) holds with equality for all group members with the same ability v_{ik} iff they receive the same prize share ϕ_{ik}^* . Note that this does not imply that these members receive a positive share of the prize.

Part ii)

Let $g''(0) = 0$, then $\frac{v_{ik}}{g''(0)} = \infty \forall k$. The optimal prize allocation satisfies the system of equations (1.17) with equality, i.e.

$$\frac{v_{i1}}{g''(x_{i1})} = \frac{v_{im}}{g''(x_{im})} \quad \forall m > 1. \quad (1.21)$$

It follows that all ik receive a positive share of the prize $\phi_{ik} > 0$. In addition, we can rewrite (1.16) as

$$\frac{g'(x_{ik})}{g''(x_{ik})\phi_{ik}} = \lambda \quad \forall k. \quad (1.22)$$

Finally, using $\sum_k^{n_i} \phi_{ik} = 1$, we can define the optimal prize allocation as

$$\phi_{ik} = \frac{\frac{g'(x_{ik})}{g''(x_{ik})}}{\sum_{ik}^{n_i} \frac{g'(x_{ik})}{g''(x_{ik})}} \quad \forall k. \quad (1.23)$$

Part iii)

Let $g''(0) > 0$. The prize allocation of all players can be derived from the relation $\sigma_i^* X^* = X_i^* = \nu_i(x_{i1}) = x_{i1} + \sum_{m>1}^{n_i} \max \left[0, g''^{-1} \left(\frac{v_{im}}{v_{i1}} g''(x_{i1}) \right) \right]$. Moreover, if

$$\frac{v_{i1}}{g''(X_i^*)} \geq \frac{v_{im}}{g''(0)} \quad \forall m > 1,$$

then it holds

$$\frac{v_{i1}}{g''(f(\frac{1-\sigma_i^*}{X^*} v_{i1}))} \geq \frac{v_{im}}{g''(0)} \quad \forall m > 1.$$

Proof of Lemma 1.2

Given a distribution of ability $\mathbf{v}_i = (v_{i1}, \dots, v_{in})$, the group i 's total effort given by (1.6) can be rewritten as

$$\sigma_i = \frac{\sum_{k=1}^{n_i} f \left(\frac{1-\sigma_i}{X} \frac{1}{n_i} v_{ik} \right)}{X}. \quad (1.24)$$

Equation (1.24) allows us to see that for each given X and \mathbf{v}_i there is a unique value of σ_i that satisfies Equation (1.6). In other words, Equation (1.6) implicitly defines σ_i as a function of \mathbf{v}_i and X ; $\sigma_i = \sigma_i(\mathbf{v}_i, X)$. The equilibrium value of X is then determined by the condition $\sum_{i=1}^N \sigma_i = 1$. Moreover, $\sigma_i(\mathbf{v}_i, X)$ is strictly decreasing and continuous in X , $\lim_{X \rightarrow \infty} \sigma_i = 0$ and $\lim_{X \rightarrow 0} \sigma_i = 1$. These properties of the share functions follow directly from Theorem 3 of Cornes and Hartley (2005). Now, we are ready to prove the Lemma examining the behaviour of $\sigma_i(\mathbf{v}_i, X)$ moving from \mathbf{v}_i to \mathbf{v}_i' when f is strictly convex.

Let σ_i^* and X^* be equilibrium values under the ability vector \mathbf{v}_i , i.e. $\sum_i^N \sigma_i^*(\mathbf{v}_i, X^*) = 1$, then:

- i) fix X^* and σ_i^* , the right hand side of (1.24) defined by $F(\mathbf{v}_i)$ is Schur-convex. Using Definition 2, a change in ability distribution such that $\mathbf{v}'_i \succ \mathbf{v}_i$ implies $F(\mathbf{v}'_i) > F(\mathbf{v}_i)$. The σ_i that solves (1.24) for the new distribution of ability, but keeping X^* fixed, is $\sigma'_i(\mathbf{v}'_i, X^*) > \sigma_i^*(\mathbf{v}_i, X^*)$. However, this is not the new equilibrium since $\sigma'_i(\mathbf{v}'_i, X^*) + \sum_{j \neq i}^N \sigma_j(\mathbf{v}_j, X^*) > 1$;
- ii) the new equilibrium total effort X^{**} satisfies $\sigma_i(\mathbf{v}'_i, X^{**}) + \sum_{j \neq i}^N \sigma_j(\mathbf{v}_j, X^{**}) = 1$. Hence, $X^{**} > X^*$ since $\sigma_i(\mathbf{v}_i, X)$ is continuous and strictly decreasing in $X \forall i$. This proves that for every group other than i the winning probability strictly falls, i.e. $\sigma_j^{**}(\mathbf{v}_j, X^{**}) < \sigma_j^*(\mathbf{v}_j, X^*) \forall j \neq i$ and $\sigma_i^{**}(\mathbf{v}'_i, X^{**}) > \sigma_i^*(\mathbf{v}_i, X^*)$;
- iii) finally, $\sigma_i^{**} > \sigma_i^*$ and $X^{**} > X^*$ imply $X_i^{**} > X_i^*$.

The same analysis holds if $\mathbf{v}'_i \prec \mathbf{v}_i$ and f is strictly concave.

Proof of Proposition 1.3.3

Part i)

Let v_{i1} be the highest ability in group i . In easy contests (f is convex) the optimal allocation $\phi_i = \phi_i^*$ rewards players as follows: $\phi_{i1} = 1$, $\phi_{im} = 0 \forall m > 1$. It follows from Lemma 1.2 that a change in ability distribution such that $\mathbf{v}'_i \succ \mathbf{v}_i$ and $v'_{i1} > v_{i1} \rightarrow \sigma_i^{**} > \sigma_i^*$. On the other hand, a change in ability distribution such that $\mathbf{v}'_i \succ \mathbf{v}_i$ but $v'_{i1} = v_{i1} \rightarrow \sigma_i^{**} = \sigma_i^*$.

Part ii-iii)

In order to prove Part ii (iii) of the proposition, it is sufficient to show that at a fixed X and σ_i the right-hand side of (1.7) is Schur-Convex (Schur-Concave). The rest of the proof follows from Lemma 1.2.

Let $v_{i1} \geq \dots \geq v_{in_i}$ and from (1.7) $F(\mathbf{v}_i) = \sum_{k=1}^{n_i} f\left(\frac{1-\sigma_i^*}{X^*} v_{ik} \phi_{ik}^*(v_{ik}, \mathbf{v}_i)\right) / X^*$. Fixing X^* and σ_i^* , then $F(\mathbf{v}_i)$ is Schur-Convex if it holds the Schur-Ostrowski criterion, $\frac{\partial F(\mathbf{v}_i)}{\partial v_{i1}} - \frac{\partial F(\mathbf{v}_i)}{\partial v_{im}} > 0 \quad \forall m > 1$. Hence, if

$$\frac{\phi_{i1}^*}{g''(x_{i1})} - \frac{\phi_{im}^*}{g''(x_{im})} + \sum_{ik}^{n_i} \frac{v_{ik} \frac{\partial \phi_{ik}^*}{\partial v_{i1}}}{g''(x_{ik})} - \sum_{ik}^{n_i} \frac{v_{ik} \frac{\partial \phi_{ik}^*}{\partial v_{im}}}{g''(x_{ik})} > 0.$$

Since $g''(0) = 0$ implies $\frac{v_{i1}}{g''(x_{i1})} = \frac{v_{im}}{g''(x_{im})}$ (see the proof of Proposition 1.3.2 part ii) and $\sum_{ik}^{n_i} \frac{\partial \phi_{ik}}{\partial v_{im}} = 0$, then $\sum_{ik}^{n_i} \frac{v_{ik} \frac{\partial \phi_{ik}^*}{\partial v_{im}}}{g''(x_{ik})} = 0$. Overall, $F(\mathbf{v}_i)$ is Schur-Convex if

$$\frac{\phi_{i1}^*}{g''(x_{i1})} > \frac{\phi_{im}^*}{g''(x_{im})}$$

Substituting ϕ_{ik}^* with Equation (1.23), we get

$$\frac{g'(x_{i1})}{g''(x_{i1})^2} > \frac{g'(x_{im})}{g''(x_{im})^2} \quad (1.25)$$

which holds when $g''(x)/g'(x) > 2g'''(x)/g''(x)$; thereby $\mathbf{v}_i' \succ \mathbf{v}_i$ implies $F(\mathbf{v}_i') > F(\mathbf{v}_i)$ and $\sigma_i'(\mathbf{v}_i', X^*) + \sum_{j>1}^N \sigma_j(\mathbf{v}_j, X^*) > 1$. Finally, part (ii-iii) of Lemma 1.2 concludes the proof.

Proof of Proposition 1.3.6

In order to maximize Equation (1.8) with respect to ϕ_{ik} , we have to solve the following Lagrangean problem:

$$L = \sum_{k=1}^{n_i} \left(\frac{1-\sigma_i^*}{X^*} (v_{ik} \phi_{ik}) \right)^{\frac{1}{\alpha-1}} + \lambda \left(1 - \sum_{k=1}^{n_i} \phi_{ik} \right),$$

which gives

$$f'_{i1}(\cdot) v_{i1} = f'_{im}(\cdot) v_{im}.$$

We can rewrite the share of the prize for any player $im \neq i1$ as a share of the prize received by player $i1$ as

$$\phi_{im} = \phi_{i1} \left(\frac{v_{im}}{v_{i1}} \right)^{\frac{1}{\alpha-2}},$$

and substituting it in the constraint gives

$$\phi_{i1} + \sum_{m=2}^{n_i} \phi_{im} \left(\frac{v_{im}}{v_{i1}} \right)^{\frac{1}{\alpha-2}} = 1.$$

A simple rearrangement defines the optimal prize allocation as

$$\phi_{i1} = \frac{v_{i1}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n_i} v_{ik}^{\frac{1}{\alpha-2}}},$$

and for every im as

$$\phi_{im}^* = \frac{v_{im}^{\frac{1}{\alpha-2}}}{\sum_{k=1}^{n_i} v_{ik}^{\frac{1}{\alpha-2}}}. \tag{1.26}$$

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Chapter 2

Inequality within Groups: Theory and Evidence

2.1 Introduction

Work is often carried out in teams that compete for prizes or bonuses. Examples of these situations include research and development races, litigation, workplace competitions and sports.

As discussed in the literature of contests, the chance of victory depends on how a team splits the prize among team members. To attain the highest probability of success, a team should distribute the prize according to the “relative effort” rule: each team member receives a share of the prize in proportion to his observable effort. In the context of groups, however, what is usually observable is the aggregate work of a team, and not the individual contributions to it.

Here, we analyse multi-team contests in which individual contributions are not observable, and players are possibly heterogeneous in their abilities. We study, both in theory and in the lab, (i) how to best allocate a prize, and (ii) how to best select members in order to help a group win.

Our analysis is relevant to organisational settings where a team manager, as well

as a social planner or a contest organiser, can shape a group’s performance by strategically manipulating the allocation of the rewards and the selection of the members. In sports, for example, it is well-known that there is a trade-off between building a team with a “superstar” versus a team with more homogeneous players.¹ In either case, a group’s performance varies with the allocation of the prize. Research departments face a similar situation when competing to attract funding. They choose whether to invest their budget on a highly skilled researcher or on several junior researchers. Further, in competitions for bonuses between sales or production departments, managers decide not only on which tasks to assign employees, but also how much to reward them based on the related responsibilities.

In the model there are two stages. In stage one, all groups strategically implement a prize allocation to maximise their probability of winning. In stage two, all players simultaneously exert an effort knowing the prize allocations implemented by all groups. Individual contributions are often unobservable in contests between groups, so we propose an approach to allocate the prize that is independent of members’ effort: before any effort is exerted, all groups commit to a prize schedule based on players’ abilities, which are public knowledge. We then study how intra-group heterogeneity in ability affect a teams’ probability of winning. Since we assume that both prize and team average ability are fixed parameters, we can naturally interpret intra-group heterogeneity as intra-group inequality.²

Our analysis reveals that the properties of the cost function themselves determine the optimal design of a team. And since the shape of the marginal cost plays a key role, we conveniently classify contests as “easy” or “hard” depending on whether

¹See the article by Rory Smith in The New York Times “By His Absence, Zlatan Ibrahimovic Makes Sweden Stronger at the World Cup” available <https://www.nytimes.com/2018/07/03/sports/world-cup/sweden-zlatan-ibrahimovic-.html>.

²In Cubel and Sanchez-Pages (2014), the authors show that there is a natural relationship between the Atkinson index of inequality and a group’s probability of winning.

the marginal cost is concave or convex respectively. Crucially, a concave (convex) marginal cost implies an increasing (decreasing) marginal productivity with respect to the prize. Regarding easy contests, we find that it is optimal to assign the entire prize to one of the most skilled group members. Regarding hard contests, we find that: if marginal productivities are unbounded above, then all players receive a positive share of the prize; if they are bounded, then the team's superstar may receive the entire prize; and if and only if teammates have the same skill, then it is optimal to reward them equally. The intuition behind these findings is easy to grasp: the first unit of the rewards for winning is always best allocated to the member with the highest marginal productivity, the team's superstar. In easy contests, the second, the third, and last unit of the prize also go to the superstar as his marginal productivity increases with the prize. In hard contests, on the contrary, the marginal productivity decreases with respect to the prize. Thus, the second unit of the rewards may go to the second highest ability player, and so on until the entire prize is allocated.

To understand how intra-group inequality in abilities affects a group's success, we must consider whether the group splits the prize equally, as often assumed in the literature, or as proposed in this paper. In the former case, a more unequal (spread-out) distribution of abilities increases a group's chance of winning in easy contests,³ while in the latter case, it increases a group's chance of winning in both easy contests and a subset of hard contests. Intuitively, when the contest is not too hard, there exists a strong complementarity between ability and rewards. If rewards are allocated in an assortative way, assigning higher shares to high ability members, then players' effort increases enough to make heterogeneous groups superior to more equal ones.

Despite the fact that many people could argue that a reduction of inequality is

³See Nitzan and Ueda (2014) and Cubel and Sanchez-Pages (2014)

always desirable, our analysis reveals that institutions are often incentivised to design unequal groups. In easy contests, regardless of whatever the source of inequality—abilities, rewards, or both—it increases a team’s performance and under additional conditions on the cost function, higher levels of within group inequality improve groups’ performances also in hard contests. Outside our model, however, inequality can affect group members’ behaviour in various ways. For instance, a high level of inequality may trigger concerns of intra-group fairness, while a lower level of it may support cooperative behaviours instead. To understand whether behavioural factors can overcome our findings on the positive effect of inequality we run a laboratory experiment.

In the lab, we conduct winner-takes-all contests à la Tullock involving two groups of two players each. While groups compete for the same prize and consists of on average equally capable players, they differ in their internal inequality. Specifically, we design four group types: fully equal, unequal in ability, unequal in prize, and unequal in both ability and prize. In order to disentangle the effects of the three inequality types we carry three treatments: 1) Treatment Ability, 2) Treatment Prize, and 3) Treatment Combination. Each treatment runs a contest between a fully equal group and a group that is either unequal in ability, in prize, or in both respectively. Importantly, we carefully choose a cost function such that unequal groups always have higher probability of winning than the equal ones.

Our empirical analysis reveals that, both in Treatment Ability and Treatment Prize, the competing groups have very similar chances of winning. In contrast to the theoretical predictions, we do not find empirical support that an unequal team in ability, or in rewards, performs better than a fully equal team. In Treatment Combination, on the other hand, the unequal group not only has considerably higher probability of winning than the equal one, but its contribution is also the highest

among all groups and across all treatments. Thus, matching high rewards to high ability players has two effects on a group’s performance: it induces a higher team effort than the equal split; and it makes the unequal group superior to the equal one, both of which are in accordance with our predictions.

Another important feature of the experiment is the analysis of players’ contributions using data on their beliefs. Contrary to their beliefs about teammates’, those about the opponents’ strategies are an important predictor of a player’s contributions. Specifically, subjects choose higher efforts when they believe the competing group does the same. Further, contributions are higher than what theory predicts, although the Nash predictions were corrected using players’ beliefs. Precisely, we call *belief adjusted deviation* (BAD) the difference between a subject’s choice and the model’s prediction adjusted to his beliefs about others. The BAD in our experiment is much larger than zero, which means subjects overbid. Finally, the BAD is correlated with subjects’ risk attitudes, and it declines with experience, suggesting that the overbidding is due to both player’s risk preference and errors.

Literature review

Much progress has been made in the study of contests since the seminal work of Tullock (1980). Regarding team contests, the literature has considered contests with different sharing rules (Nitzan, 1991a,b), group sizes (Esteban and Ray, 2001; Nitzan and Ueda, 2011), heterogeneous players (Baik, 2008; Nitzan and Ueda, 2018; Choi et al., 2016) and timings of the choices (Balart et al., 2018). These papers employ one of the following prize allocations among winning group members: the egalitarian rule, the relative effort rule, or any linear combination of the two; of which the relative effort rule better incentivises groups.⁴ Indeed, its use eliminates the free-riding issue

⁴See Flamand et al. (2015) for a survey.

by putting teammates in competition for the appropriation of the prize. However, it is contingent on ex-post individual efforts, which are not always observable.

Nitzan and Ueda (2014) and Cubel and Sanchez-Pages (2014) are the few models studying prize allocations that are independent of players' effort provisions. Although assuming symmetry among players, they find that the equal split maximises a team's performance only if a player's marginal costs of effort increases rapidly enough, while unequal division are otherwise more efficient. In an incomplete information contest for a pure private prize, Trevisan (2020) shows that when players are heterogeneous, then unequal allocations can improve a team's effort even though marginal costs increase very quickly. In this paper, by contrast, we study complete information contests, where the prize is a mix of a public and a private good (Esteban and Ray, 2001). Further, we conduct a lab analysis related to implications of intra-group inequality.

The literature on contests has also dedicated much attention to empirically test models' predictions, especially those regarding sharing rules (Gunnthorsdottir and Rapoport, 2006; Amaldoss et al., 2000; Kugler et al., 2010), team sizes (Abbink et al., 2010; Ahn et al., 2011), endowments (Heap et al., 2015), alliance formations (Herbst et al., 2015), and power differentials (Bhattacharya, 2016). Most of the experiments design groups with symmetric group members. Exceptions are in Sheremeta (2011), where groups have a stronger member, and in Brookins et al. (2015a), where all players differ in their cost of contributing. However, these papers do not relate to our set-up since they test predictions on the use of different success functions and different sorting of players, rather than investigating inequality issues. Both in theory and in the lab, non-incentivised types of heterogeneity are studied in Konrad and Morath (2019). The authors model a dynamic contest in which contestants possibly differ only in behavioural motives that go beyond the payoff maximisation. Learning

about others' motives and self-selection have possible implications on players' effort escalation. The corresponding experimental set-up provides evidence for such heterogeneous motives, for self-selection and for effort escalations. Similar to our study, they also find a persistent and positive correlation between subjects' effort and their beliefs about opponents' efforts.

The experimental paper most similar to ours is Fallucchi et al. (2019), which carries contests involving two groups of three players each. Groups can be of two types: fully equal, or unequal in ability. Depending on the treatment, they compete either against another group of their same type, or against a different one. The authors' main finding is that the highest total effort is obtained in a competition between two unequal groups. The authors also run a treatment involving a fully equal group and a group unequal in ability, which is comparable to our Treatment Ability. Here, they don't find substantial differences between the two groups' chances of winning, a result in accordance to ours. Despite this similarity, the two papers' experimental designs differ substantially. We randomly rematch players every round whilst they employ a partner-matching protocol.⁵ We use a convex cost function instead of a linear one. And we further analyse the effects on team effectiveness of three types of internal inequality, rather than studying groups' behaviour under different matching of groups. But more importantly, to the best of our knowledge we are the first paper to provide a theory and empirical evidence of the positive effect of inequality on group effectiveness.

Section 2.2 presents our model of contests; Section 2.3 presents the experimental findings; Section 2.4 concludes.

⁵In other words, we try to avoid any cooperative behaviour that may occur in a partner-matching protocol given the repeated interaction between players.

2.2 The Model

In this section, we develop our model where groups compete to obtain a prize. The model is dynamic with two stages: a prize allocation stage, and a contest stage. In the former, all groups simultaneously choose a prize allocation to maximise their chances of winning. The allocation is independent of individual's efforts. In the latter, all players simultaneously exert an effort knowing the prize allocations implemented by all groups. The solution concept is the Subgame perfect Nash equilibrium in pure strategies.

2.2.1 The preliminaries

We consider N groups. The i -th group is composed of n_i risk-neutral members who are indexed by $ik = (i1, \dots, in_i)$. The prize is a mixture of a public good part, $P \geq 0$, and a private one, $\Phi > 0$. The private part is shared among group i 's members according to the allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ s.t. $\sum_{k=1}^{n_i} \phi_{ik} = \Phi_i$ and $\phi_{ik} \geq 0 \forall k$. If group i wins the contests, then each of its members receives a reward of $P + \phi_{ik}$. All players ik exert an effort $x_{ik} \geq 0$ at a cost $v_{ik}^{-1}g(x_{ik})$, where $v_{ik} \in (0, \infty)$ is the (possible) heterogeneous ability parameter.⁶ Group i 's total effort is $X_i = \sum_{k=1}^{n_i} x_{ik}$, and group i 's probability of winning is given by the Tullock success function $\sigma_i = X_i/X$, where $X = \sum_i^N X_i$. Overall, for an arbitrary prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i}) \forall i$, the expected payoff of player ik is given by

$$\pi_{ik} = \frac{X_i}{X}(P_i + \phi_{ik}) - \frac{g(x_{ik})}{v_{ik}}. \quad (2.1)$$

The case of linear costs, $g(x) = x$, is studied in Baik (2008). The author shows

⁶This approach to define heterogeneity is commonly used in the literature of contests, see for example Ryvkin (2011, 2013), Brookins et al. (2015b), Nitzan and Ueda (2018) and Trevisan (2020).

that only one member in each group exerts a positive effort, the superstar. In our sequential game, linear costs would make the analysis trivial: as a result of the stark intra-group free-riding, the allocation that maximises a group's chances of winning assigns the entire prize to the team's superstar. To avoid this situation, we assume that the cost function $g(x)$ is strictly convex. Under the following additional conditions, this guarantees that all players receiving a prize exert a positive effort.

Assumption 1 i) $g(0) = 0$; ii) $g'(0) = 0$; iii) $g'(x) > 0$ for all $x > 0$; iv) $g''(x) > 0$ for all $x > 0$; v) $g'''(x)$ exists for all $x > 0$. Since g is monotonic and continuous, it has a well-defined inverse function, $f = (g')^{-1}$.

Under Assumption 1, the first-order condition of π_{ik} , subject to $x_{ik} \geq 0$ is necessary and sufficient for player ik 's best response:

$$\frac{X - X_i}{X^2}(P_i + \phi_{ik})v_{ik} = g'(x_{ik}). \quad (2.2)$$

The contest stage has a unique pure strategy Nash equilibrium in which all groups always exert a strictly positive effort.

Lemma 2.1. *Given Assumption 1, the contest between groups has a unique equilibrium effort $X_i^* \forall i$ in pure strategies. The equilibrium levels of effort $X_i^* = \sum_k^{n_i} x_{ik}^*$ satisfies the system of equations (2) with equality and defines the group i 's effort as*

$$X_i^* = \sum_{k=1}^{n_i} x_{ik}^* = \sum_{k=1}^{n_i} f\left(\frac{X^* - X_i^*}{(X^*)^2}(P_i + \phi_{ik})v_{ik}\right). \quad (2.3)$$

Proof. Recalling that $\sigma_i = X_i/X$, we can rewrite Equation (2.3) as

$$\sigma_i = \sum_{k=1}^{n_i} f\left(\frac{1 - \sigma_i}{X}(P_i + \phi_{ik})v_{ik}\right) / X. \quad (2.4)$$

Equation (2.4) implicitly defines a group i 's probability of winning as a function of the aggregate effort X , $\sigma_i = s_i(X)$. The equilibrium value of X is determined by the condition $\sum_{i=1}^N s_i(X) = 1$. Note that the left-hand side of (2.4) exceeds the right at $\sigma_i = 1$. Furthermore, the right-hand side is decreasing on σ_i , which implies that there is a unique σ_i that solves (2.4) for any $X > 0$. Finally, because $\sigma_i = s_i(X)$ is strictly decreasing and continuous in X for all i , $\lim_{X \rightarrow \infty} s_i(X) = 0$ and $\lim_{X \rightarrow 0} s_i(X) = 1$,⁷ then it should be clear by the intermediate value theorem that there is only one equilibrium aggregate effort $0 < X^* < \infty$ such that $\sum_{i=1}^N s_i(X^*) = 1$. Finally, the equilibrium aggregate X^* and probability of winning σ_i^* define the groups' efforts as $X_i^* = \sigma_i^* X^* \forall i$. \square

Hereafter, as in Trevisan (2020), we conveniently refer to two types of contests: “easy” and “hard”.

Definition 1. A contest is “easy” when f is strictly convex (g' concave). Conversely, a contest is “hard” when f is strictly concave (g' convex).

2.2.2 The prize allocation

As discussed in the introduction, a prize allocation that rewards players according to their relative effort eliminates the free-riding problem. However, it requires observation of individual contributions. As what is usually observable is the final work of a team, and not the individual contributions to it, we consider prize allocations that are not contingent on players' ex-post efforts. Formally, all groups $i = 1, \dots, N$

⁷Function $s_i(X)$ is known as the “share function” and its properties follow directly from Cornes and Hartley (2005).

simultaneously choose a prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$ in order to maximise their probability of winning $\sigma_i = X_i/X$.⁸ Thus, the group i 's objective function is given by

$$\begin{aligned} \phi_i &\in \operatorname{argmax} \sigma_i \\ \text{s.t. } \sum_k^{n_i} \phi_{ik} &= \Phi_i, \phi_{ik} \geq 0 \quad \forall k. \end{aligned} \tag{2.5}$$

If we find a profile of prize allocations $(\phi_i^*, \dots, \phi_N^*)$ that solves (2.5) for all i and all players maximise their expected payoff, then we can state that it is a Subgame Perfect Nash Equilibrium in pure strategies. As established by next proposition, the number of equilibrium prize allocations depends on whether the contest is easy or hard, and on group members' ability.⁹

Proposition 2.2.1. *Suppose that the contest is easy, then the model has $\prod_{i=1}^{\infty} n_i^h$ Subgame Perfect Equilibria, where n_i^h is the number of group members with the highest ability in group i . Suppose that the contest is hard, then the model has a unique Subgame Perfect Equilibrium. In equilibrium, the properties of group i 's allocation $\phi_i^* = (\phi_{i1}^*, \dots, \phi_{in_i}^*)$ can be summarised as follows:*

- i) in easy contests, ϕ_i^* rewards the entire prize to one of the n_i^h members;*
- ii) in hard contests, ϕ_i^* rewards members with the same ability equally, $v_{ik} = v_{im}$ implies $\phi_{ik}^* = \phi_{im}^*$;*

⁸This is a common situation assuming that the prize division is imposed by a third subject, whose compensation is aligned with the results of the group. Examples include organisations that use contests to boost workers productivity and retail firms that set-up monetary reward contests for sales departments during periods with a peak in the demand for goods. Furthermore, in sports competitions managers face the task of dividing the prize among the winning members.

⁹In Appendix A, we show that an allocation that maximises a group's probability of winning also maximises the group effort. In other words, under the equilibrium profile of prize allocations $(\phi_i^*, \dots, \phi_N^*)$ no group has an incentive to deviate by implementing a different allocation rule neither to increase its probability of winning nor to increase its effort.

iii) in hard contests where $P_i > 0$ and/or $g''(0) > 0$, ϕ_i^* rewards the highest ability members for any equilibrium efforts X^* , $\phi_{ik}^* > 0 \forall X^*$ iff $v_{ik} = \max[v]$.

The interpretation of the above proposition is easy to grasp if we interpret the vector of players' effort decisions, Equation (2.3), as a vector of effort production functions. When the contest is easy (f is convex), then players' marginal productivity of effort increases in the prize share ϕ_{ik} . So, the group probability of winning is maximised allocating the entire prize to the most skilled player. On the other hand, when the contest is hard (f is concave), then the players' marginal productivity of effort decreases in the prize share ϕ_{ik} possibly leading to a more equal distribution of the prize. However, it may still be the case that a high ability player is more productive (at the margin) than his teammates even if he receives the entire prize.

Despite the results of the proposition resemble those of standard constrained optimisation, it is important to keep in mind that the prize allocations are chosen strategically by all groups. Furthermore, the use of such allocation does not merely maximise a group effectiveness for a given distribution of ability, but it also has implications on the optimal team composition as discussed in the following section.

2.2.3 The effects of intra-group inequality

Before moving to the experimental analysis, we revise recent results on the effects of intra-group inequality in contests. This approach is convenient to highlight how such findings relate to the allocative scheme proposed in the previous section and to introduce the hypothesis we test experimentally.

For simplicity, suppose there are two groups, A and B, formed by two players each. Groups are *on average* equally skilled, but abilities in group B are distributed more

unequally (in the sense of Lorenz’s distribution) than in group A.¹⁰ For example, group B has one member more skilled than the other while group A’s players are symmetric.

In this situation, under which conditions is group B more effective than A? The answer to that question can be found in Nitzan and Ueda (2014) and Cubel and Sanchez-Pages (2014). The authors show that the greater the inequality across one dimensional group characteristics, the higher (lower) a group’s efficiency if the contest is easy (hard). In our example, if members in group A and group B split the prize equally, but group B is more unequal than A on the ability dimension, then B has higher (lower) chances of winning in easy (hard) contests. This result can be extended to the case in which members of the two groups are symmetric in ability, but group B is more unequal than A on the reward dimension.¹¹

Suppose now that group B changes its allocation from the egalitarian rule to a more efficient one. How does this affect its effectiveness? Clearly, group B’s effectiveness must increase independently on the type of contest. Furthermore, as shown in Trevisan (2020), if groups implement the optimal allocation, then groups unequal in ability are stronger than more equal ones.¹²

Overall, the analysis on contests reveals that organisations are often incentivised to design unequal groups. In easy contests, regardless of the source of inequality—abilities, rewards, or both—it increases a team’s performance. Under additional

¹⁰Note that, in our setting, a higher within group heterogeneity (mean-preserving spread) can be viewed as a higher within group inequality (in the sense of Lorenz dominance). Similarly, a more unequal distribution of the reward can be interpreted as a (reverse) Pigou-Dalton transfer, i.e. a commitment to transfer the gain from one member to another conditional on winning the contest. This relation between heterogeneity and inequality allow us to straightforwardly link our results to the inequality and welfare literature.

¹¹A model with multiplicative heterogeneity is mathematically equivalent to a model with heterogeneity in rewards.

¹²For a pure private prize, Proposition 3.3 in Trevisan (2020) extends to our setting with complete information.

conditions on the cost function, the same occurs in hard contests. Outside our model, however, a high level inequality can trigger concerns of fairness, while a lower level of it may support cooperative behaviours instead. To understand whether behavioural factors can overcome the theoretical findings on the positive effect of inequality we run a laboratory experiment. In the experiment (see below) we create a competition between groups, very similar to the example provided above, to empirically test the following theoretical predictions about easy contests: the greater the within group inequality in ability, rewards, or their efficient combination, the higher the group effort and chances of winning.

2.3 The experiment

We consider a contest between two groups, A and B, of two players each. The winning group receives a total prize of $\Phi = 1000$ while the losing group receives nothing. The winning members share the prize either equally, a 50-50 split, or unequally, a 75-25 split. The players' cost function is $g(x)/v_{ik} = 10x^{1.2}/v_{ik}$, i.e. the contest is "easy". There are three types of players: L, M, H - with ability parameter $v_L = 1, v_M = 2, v_H = 3$ representing low, medium and high ability respectively.¹³ Across all three treatments, group A is the equal group as it consists of two M players that equally share the prize. On the contrary, group B is the unequal group.

We implement a total of three between-subjects treatments. In Treatment Ability, which studies the inequality in ability, group A=(M, M; 50, 50) competes against B=(H, L; 50, 50). In Treatment Prize, which studies the inequality in reward, group A=(M, M; 50, 50) competes against B=(M, M; 75, 25). As seen from Table 2.1, players' equilibrium effort levels are identical among the two treatments as we created

¹³To avoid framing, we used X, Y, Z in the experiments.

an equivalent level of inequality. In Treatment Combination, which studies the efficient combination of the two types of inequality, group A=(M, M; 50, 50) competes against B=(H, L; 75, 25). As seen from Table 2.1, assigning higher rewards to the H player predicts the highest group effort and the probability of winning among all treatments. Thus, the following hypotheses are formulated based on the predictions in table 2.1.

Hypothesis 1.

- a) In Treatment Ability, group B contributes more than group A;
- b) In Treatment Prize, group B contributes more than group A;
- c) In Treatment Combination, group B contributes more than group A;

Hypothesis 2.

In Treatment Combination, group B shows the highest contributions and probability of winning among all groups and treatments.

	Treatment Ability			Treatment Prize			Treatment Combination		
	group A	group B		group A	group B		group A	group B	
Type	M	L H		M	L H		M	L H	
Individual Costs	52	0 100		52	0 150		50	0 144	
Individual Payoff	470	578 478		470	439 567		430	459 633	
Group Payoff	940	1056		940	1006		860	1092	
Individual Effort	7	0.1 17		7	0.1 17		6.6	0 23.1	
Group Effort	14	17.1		14	17.1		13.2	23.1	
P. of Winning	44.4	55.6		44.4	55.6		36.4	63.6	

Table 2.1: Theoretical predictions

Note: Individual payoff and group payoff include the 300 endowment points for each subject.

2.3.1 Design and procedure

The experiment was conducted at the BLUE lab at the University of Edinburgh and programmed in z-Tree (Fischbacher, 2007). We ran a total of 16 sessions with 12 or 16 subjects per session. In total, we recruited 168 subjects from university subject pool. Subjects were allowed to participate in only one session. They earned an average of \$12.9, including a show-up fee of \$3, for a session lasting approximately 75 minutes. After finishing the main part of the experiment, we conducted an incentivised 12-question IQ test. Then, we elicited the subjects' risk preferences with real incentives using the Holt and Laury (2002)'s method. Finally, we surveyed subjects with personality questions and basic information such as gender and age. The printed instructions were distributed and read aloud by an experimenter to assist understanding. The instructions can be found in Appendix B.

The contest part of the experiment lasted a total of 30 rounds. At the beginning of the experiment, subjects were randomly assigned as type L, M, H and learned the allocation rule. Subjects were informed that their role and the prize division stayed the same during the entire experiment. To eliminate the repeated game effect, we implemented a random matching mechanism. In each new round, subjects were randomly matched with another subject to form a new group, and to compete with another group formed in the same way.

The groups competed for a prize worth 1,000 points in a Tullock way. In each round, all subjects received an endowment of 300 points, which they could either use to invest in the group account contributing an effort $x_i \in [0, 50]$,¹⁴ or save for personal payoff. In order to reduce the ceiling effect of the endowment, subjects

¹⁴Equivalently, in the instruction we used the terminology "lottery tickets" instead of "effort" as in Chowdhury et al. (2019).

were allowed to invest beyond their endowment.¹⁵ Notably, the endowment and contribution limit were carefully selected to construct a fair competition. Specifically, for all treatments, subjects could learn that their group has an approximately 50% chance of winning if all players, regardless of their types and allocation rules, invested either the minimum or maximum of the contribution limit, or they used all their endowments as contributions.

To investigate the role of beliefs, we incentivised subjects to submit their predictions about their own group's and the opponent group's contribution (while making their own contribution decisions). In every round, subjects received a reward of 50 points for each correct prediction. In order to reduce the curiosity effect, at the end of each round, we provided each player with feedback that includes the total contributions of both groups, the probability of winning of both groups, the winning group and the payoff (see Appendix B). At the very end of the experiment, subjects received real payment from 5 randomly selected rounds (Brookins et al., 2015a).

2.3.2 Group level results

In this section, we describe the group and contest level findings. Table 2.2 reports the summary statistics of the efforts and winning probabilities in comparison to the theoretically predicted values. It shows that, on average, unequal groups contribute as much as the equal ones in Treatment Ability and Prize. On the other hand, they outperform the equal group in the Treatment Combination.

On average, group contributions goes against Hypotheses 1a, and 1b, i.e. the existence of competitive advantage of within-group inequality in either abilities or

¹⁵Subjects were informed that at the end of the experiment they received at least the show-up fee. However, they were warned that they could receive a negative payoff for the contest part, which reduces the strictly positive payoffs in the following parts of the experiment.

rewards, as efforts in Treatment Ability and 2 are similar across groups. On the other hand, there is evidence in support of Hypothesis 1c, i.e. the positive effect of an effective combination of the two inequalities, as group effort is 30% higher for group B in Treatment Combination.

	Treatment Ability			Treatment Prize			Treatment Combination		
	group A	group B		group A	group B		group A	group B	
Type	M	L	H	M	L	H	M	L	H
Individual Costs	176 (52)	160 (0)	175 (100)	148 (52)	91 (0)	249 (150)	143 (50)	115 (0)	244 (144)
Individual Payoff	382 (470)	382 (578)	367 (478)	397 (470)	336 (439)	434 (567)	368 (430)	329 (459)	488 (633)
Group Payoff	764 (940)	749 (1056)		794 (940)	770 (1006)		736 (860)	817 (1092)	
Individual Effort	18.6 (7)	9.3 (0.1)	26.2 (17)	16.2 (7)	10.3 (0.1)	25 (17)	15.6 (6.6)	7.1 (0)	35.1 (23.1)
Group Effort	37.2 (14)	35.5 (17.1)		32.4 (14)	35.3 (17.1)		31.2 (13.2)	42.2 (23.1)	
P. of Winning	51.5 (44.4)	48.5 (55.6)		48.9 (44.4)	51.1 (55.6)		42.4 (36.4)	57.6 (63.6)	

Table 2.2: Comparison statistics with experimental results and predicted values

Note: Theoretical predictions are in parentheses. Individual payoff and group payoff include the 300 endowment points for each subject.

In Table 2.3, we show the results of the multilevel linear mixed-effects regressions. The regressions investigate how the total contribution of the unequal groups differ from the equal groups across treatments, taking into account the inter-dependency of observations in the same experimental sessions and the same individuals. Specifically, although not statistically significant, the unequal groups show a lower contribution than the equal groups in Treatment Ability, a result following Fallucchi et al. (2019). Similarly, the unequal groups have a slightly and insignificantly higher contribution than equal groups in Treatment Prize. On the other hand, the unequal groups demonstrate a significantly higher total contribution in Treatment Combination.¹⁶

Result 2.1. *Compared with equal groups, unequal groups either in the ability or reward do not demonstrate higher contributions. Conversely, unequal groups with an*

¹⁶We also conducted a series of Kruskal-Wallis test on groups B's (the unequal groups) contributions by including all treatments, Treatment Combination and Treatment Combination. The results are all statistically significant ($p < 0.01$) meaning that the group B contribution is highest in Treatment Combination.

efficient combination of the two types of inequality outperform. The unequal groups in Treatment Combination show the highest level of contribution.

Although the unequal groups in Treatment Combination have a probability of winning lower than what theory predicts, we argue that our experimental observation is a conservative estimation of the effect of inequality. First, as shown in Figure 2.1, 28.6% of the decisions of H types in Treatment Combination equal the upper limit of their contribution. Second, our experimental setup is likely to promote overbidding, which levels out theoretical differences between groups' probability of winning since we adopted the probabilistic rule of reward allocation rather than the proportional rule. A probabilistic allocation makes contributions a riskier investment, thus promoting overbidding due to uncertainty, for example, see Chowdhury et al. (2014) and Masiliūnas (2019).

Dependent variable: group contribution			
	Treatment Ability	Treatment Prize	Treatment Combination
Heterogeneous group	-1.508 (2.558)	3.674 (1.92)	11.71*** (2.294)
Period	-0.327*** (0.0564)	-0.0528 (0.0498)	-0.248*** (0.053)
Constant	44.17*** (5.02)	29.10*** (3.485)	22.92*** (3.72)
Observations	780	900	840

Table 2.3: group contribution difference between equal and unequal groups

Note: Multilevel linear mixed-effects models using random intercepts for experimental sessions and individual subjects. Heterogeneous group is a dummy variable with the equal group being 0 and the unequal group being 1. Numbers in parentheses indicate standard errors. Significance levels * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

2.3.3 Individual-level results

Overbidding

As presented in Table 2.2, it is evident that there is a substantial overbidding compared to the Nash prediction. The overbidding is ubiquitous and similar across treatments and types. The average overbidding for Treatment Ability, Treatment Prize and Treatment Combination is 10, 9.1 and 9.4 respectively. The average overbidding for type L, type M and type H is 8.8, 9.9 and 9.7 respectively.¹⁷

Result 2.2.

- a) There is a substantial overbidding by all types in all treatments compared to the Nash Equilibrium predictions.*
- b) On average, the magnitude of overbidding is similar across all treatments and types.*

Distribution of effort

In Figure 2.1, we present the distribution of players' efforts, which are clearly dispersed, or overspread (Chowdhury et al., 2014), for all types in all treatments. However, there seems to be a first-order stochastic dominance in effort choices between types in all treatments. That is, consistent with the theory and in terms of distribution, H type subjects exert higher effort than M type subjects, and M type subjects exert higher effort than L type subjects. The figure also shows that zero contributions for L type subjects, though predicted by the model, are not commonly

¹⁷Overbidding in contest experiments has been found and addressed by many existing studies (Brookins et al., 2015a; Abbink et al., 2010; Ahn et al., 2011; Fallucchi et al., 2019)

observed.¹⁸ Overall, we can state that subjects were responsive to the types they were assigned.¹⁹

Result 2.3. *The contribution decisions of the subjects are responsive to their types, i.e. $H \text{ type} > M \text{ type} > L \text{ type}$.*

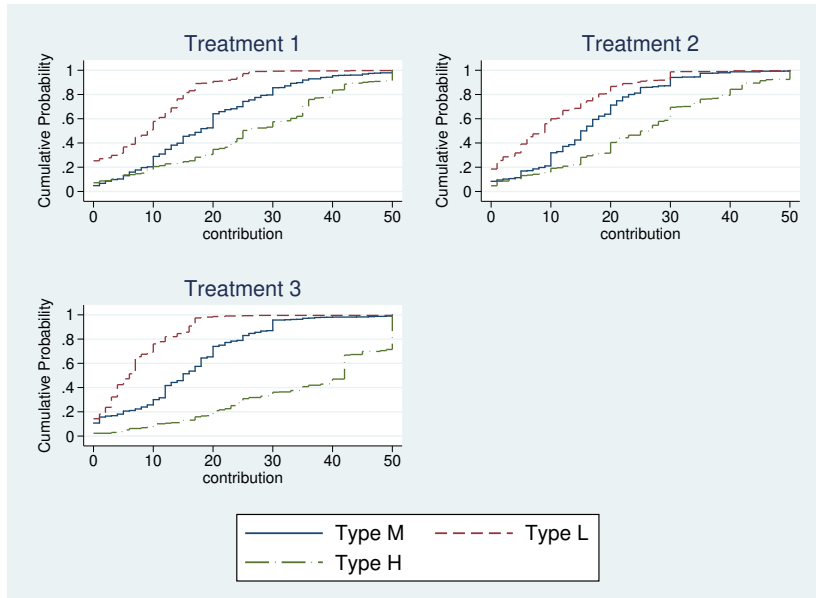


Figure 2.1: Effort dispersion by treatment

Beliefs

Table 2.4 presents a multi-level mixed effect Tobit regression controlling for sessions and individuals.²⁰ It allows us to account for the potential dependence of the contribution decisions within each session and by each individual. In all models, we

¹⁸In addition, the proportion of zero contributions is similar across treatments, while the proportion of maximum contribution choices are more frequent in Treatment Combination.

¹⁹See also Table 2.4.

²⁰We use a Tobit model because we observe a large fraction of the decisions made by type H players equal to the upper limit of the contribution, 50. For example, in Treatment Combination, 28.6% of the contribution choices made by type H players are 50.

include the variables *L type* and *H type* to capture the effects of different costs and prize share, the variable *period* to capture the potential trend over time and the variable *L contribution* to control for path dependency. It is evident that players respond to their type since *L type*'s parameter is negative, *H type*'s parameter is positive, and both are significant in all models. The *period*'s parameter is negative and significant in all models across Treatment Ability and Treatment Combination, which suggests that subjects can potentially learn to reduce overbidding over time.²¹

In Model 2, we include the predictions submitted by subjects to investigate how strategic considerations affect contribution decisions. The results indicate that subjects are strongly and positively responsive to the total contributions of their opponents, as individuals choose to contribute more if they believe their opponent group has a greater total contribution. On the other hand, we don't find any significant relationship between individual contribution and the predicted contribution of their peer. In other words, we do not find evidence of punishing free-riding or rewarding cooperative behaviours within a group, but competitive pattern between groups.

Result 2.4. *Subjects' contributions are positively affected by the expectation of their opponent group's contribution levels, but not correlated with the belief on their peer group member's contribution.*

Finally, model 3 controls for personal characteristics including the variables *Female*, *IQ score* and *Risk-seeking*. In our experiment, we don't find any gender difference in terms of contribution decisions.²² IQ tests do not have significant predicting power, while the risk-seeking parameter, which is measured by the Holt and Laury (2002)'s lottery method, seem to be positively correlated with contribution, a result confirmed

²¹Declining contributions are consistent with many prior experiments (Brookins et al., 2015a; Cason et al., 2012, 2017; Fallucchi et al., 2019) Notably however, significant overbidding still presents at the end of 30 rounds.

²²Previous studies provide mixed evidence (Heap et al., 2015; Baik et al., 2019).

Dependent Variable: individual contributions									
	Treatment Ability			Treatment Prize			Treatment Combination		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
L.contribution	0.387*** (0.025)	0.360*** (0.025)	0.360*** (0.025)	0.281*** (0.024)	0.271*** (0.024)	0.271*** (0.024)	0.352*** (0.026)	0.335*** -(0.025)	0.374*** (0.028)
Period	-0.116*** (0.023)	-0.0880*** (0.023)	-0.0881*** (0.023)	-0.020 (0.021)	-0.026 (0.021)	-0.026 (0.021)	-0.0816*** (0.022)	-0.0699** (0.022)	-0.0759** (0.024)
L-Type	-5.901** (1.925)	-5.964** (1.855)	-6.123*** (1.796)	-4.061* (1.922)	-3.884* (1.936)	-3.984* (2.006)	-5.527** (1.991)	-5.889** (2.107)	-6.627** (2.166)
H.Type	5.029** (1.920)	4.899** (1.862)	5.740** (1.885)	7.190*** (1.929)	7.299*** (1.936)	7.091*** (1.912)	14.91*** (2.050)	16.41*** (2.102)	15.04*** (2.064)
Guess_peer		-0.010 (0.024)	-0.010 (0.024)		-0.024 (0.024)	-0.024 (0.024)		0.047 (0.024)	0.039 (0.026)
Guess_other		0.134*** (0.016)	0.133*** (0.016)		0.105*** (0.018)	0.105*** (0.018)		0.0906*** (0.017)	0.0758*** (0.018)
Female			-0.151 (1.525)			-0.578 (1.586)			-0.764 (1.647)
Risk-seeking			0.806* (0.398)			0.760 (0.555)			1.061* (0.478)
IQ score			0.511 (0.468)			-0.044 (0.565)			0.081 (0.516)
Constant	13.22*** (1.277)	8.402*** (1.389)	-0.118 (4.994)	11.88*** (1.217)	8.863*** (1.371)	6.139 (6.348)	11.27*** (1.274)	6.749*** (1.489)	1.462 (5.995)
Observations	1560	1560	1560	1800	1800	1800	1680	1680	1320 ^a

Table 2.4: Individual contribution multi-level Tobit regression

Note: Multilevel Tobit models using random intercepts for experimental sessions and individual subjects. The upper limit of the Tobit model is 50. L.contribution is the individual contribution of the previous round. L-Type and H-type are categorical variables standing for the low ability/reward subjects and high ability/reward subjects respectively. The omitted category is the medium type. Guess_other stands for the expectation about the opponent group's total contribution. Guess_peer stands for the expectation about the other group members' individual contribution. Female is a dummy variable with female subjects being 1. Risk-seeking ranges from 1 to 10 with higher values indicating more risk-seeking attitudes. IQ score is the total No. of correct answers from 12 questions. Numbers in parentheses indicate standard errors. Significance levels * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

^aDue to a technical problem, we lost the information on the very last stage of the experiment including gender and age for one session (12 subjects). The rest of the session, including final payments, was not affected.

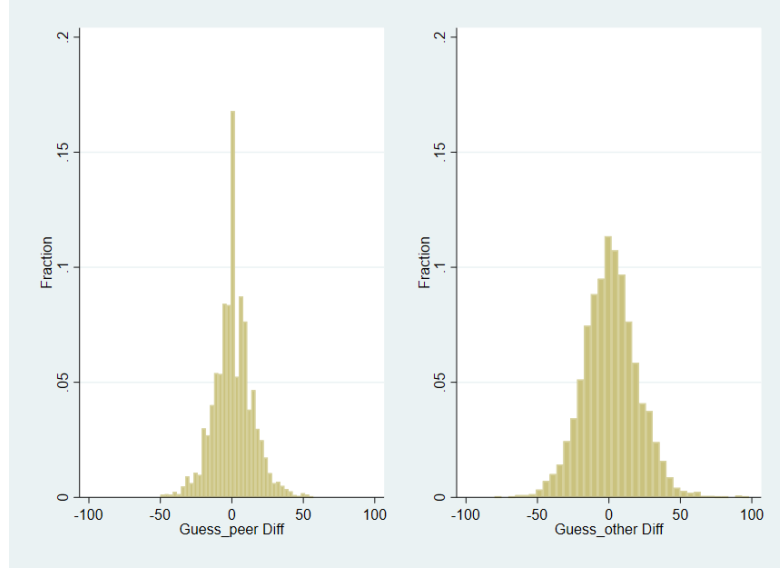


Figure 2.2: The gap between prediction and actual outcome

Note: Guess_peer difference is calculated by the prediction elicited minus the actual contributions by the subject's peer group member. Guess_other difference is calculated by the prediction elicited minus the actual contributions by the subject's opponent group total contributions.

in many other studies (Sheremeta, 2011; Mago et al., 2016). Intuitively, contributing zero is a safe choice as it guarantees a secure payoff of 300 (endowment) points. On the other hand, a strictly positive contribution is a risky choice since it involves uncertainty on the outcome of the competition.

2.3.4 Belief adjusted deviation

As it often occurs in the experimental literature on contests, we find that subjects' behaviour deviates from the Nash predictions. Broadly speaking, behavioural deviations could be a result of two grounds: strategic uncertainty, e.g. subjects fail to correctly predict other people's actions, and personal characteristics, e.g. latter

social preference and cognitive limitation.²³ The research challenge is to identify the significance of latter in the presence of the former. In order to do this, we introduce the use of the *Belief Adjusted Deviation (BAD)* constructed as follows. We collect players' beliefs about others' strategies to derive the contribution that, according to the theory, maximises their expected payoff.²⁴ Then, we define the difference between the observed contribution and the belief-adjusted optimal contribution as BAD.

$$\text{BAD} = \text{actual contribution} - \text{belief-adjusted optimum},$$

which represents the behavioural deviations free from strategic concerns.²⁵

By investigating the determinants of BAD we can identify the significance of the personal characteristics. We focus on the significance of four potential factors: competitiveness, risk attitude, cognitive ability and gender. Competitiveness is measured by a score based on four personality questions from Duffy and Kornienko (2010).²⁶ Risk preference collected through Holt and Laury (2002). Cognitive abilities are measured by an incentivised Raven matrix.

Figure 2.2 shows how subjects' beliefs are distributed. It is evident that subjects on average hold unbiased predictions about their peer's and opponent's contribution choices. The distribution of the difference peaks around zero²⁷ and it is symmetric around the mean. However, the distributions are quite dispersed, suggesting

²³See Sheremeta (2018) for a list of potential explanations on overbidding.

²⁴The belief-adjusted optimal contribution can be derived by replacing the player's beliefs about others in Equation (2.2).

²⁵We treat BAD as a directional difference instead of an absolute difference because its distribution consists of both errors and preferences.

²⁶The competition score ranges from -10 to 10, with higher values suggesting competition seeking. We provide two for-competition questions and two against-competition questions, and subjects could choose on a scale of 1 to 6 from strongly disagree to strongly agree. For for-competition questions, we generate a score from 0 to 5, and -5 to 0 for against-competition questions. The competitive score is the summed score of four questions.

²⁷The mean is 0.70 and 1.1 for peer and opponent respectively.

heterogeneity in the prediction capability in a strategically uncertain environment. On the other hand, Figure 2.3 shows that BAD is asymmetrically distributed. In all treatments, the distributions of BAD are similar and negatively skewed with a positive mean. It should also be noted that there are peaks around zero across all treatments, suggesting that some subjects indeed maximise their expected payoff according to their beliefs. The significant positive value of the BAD, together with unbiased predictions, indicates that the systematic overbidding in our experiment is mainly driven by personal characteristics rather than strategic concerns.

To further specify the determinants of BAD, we conduct multilevel mixed effects regressions, which are presented in Table 2.5. We find that, BAD in Treatment Prize cannot be explained by our model. On the other hand, it can be explained by our model and it shows very similar patterns in Treatment Ability and Treatment Combination. First, the variable period has a negative impact on BAD, thus indicating that subjects learn to bid optimally over repetition. Second, risk preference has an economically and statistically positive impact on BAD. Finally, we do not find a correlation between competitive personality or cognitive score and BAD.²⁸

Result 2.5. *The BAD in our experiment is positive and diminishes over time. It is positively correlated with the risk-seeking. On the other hand, competitiveness and cognitive skills do not have significant impact on BAD.*

2.4 Conclusions

Organisations often use contests to increase the competition within the workplace and then boost workers' productivity. For instance, team managers, whose compen-

²⁸BAD is type specific. Compared to the M type, L type subjects show significantly less BAD. It seems that subjects with a disadvantageous role are less likely to overbid.

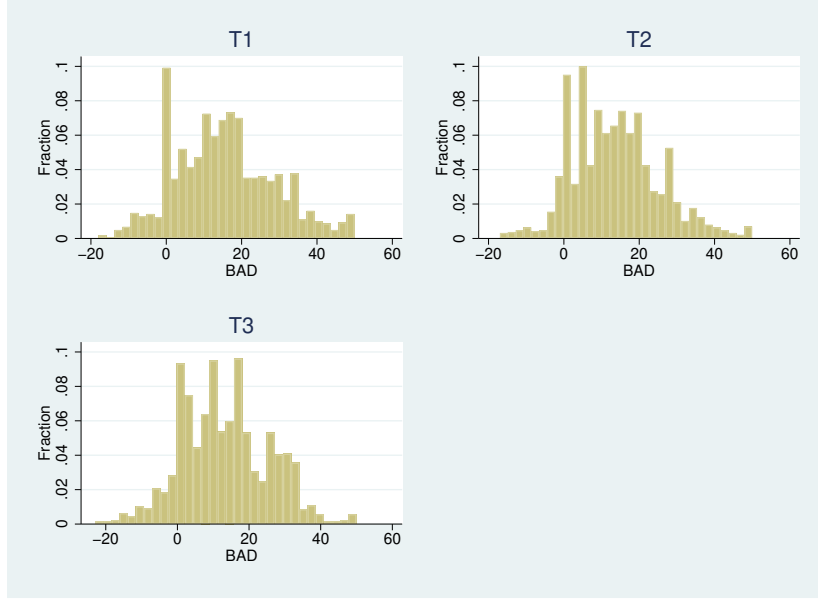


Figure 2.3: The distribution of BAD

Note: For a more detailed distribution of BAD across treatments and types, see the Appendix C

sation is usually aligned with the results of their group, often face the non-trivial task of dividing the prize among heterogeneous members. This task is particularly challenging when it is impossible to observe individual level contributions. Thus, for such competitive environments, we provide a mechanism of prize division that does not require observing individuals' contributions and that maximises a group's effectiveness. Our main findings are the following: in easy contests it is optimal to allocate the entire prize to the most able group members, while in hard contests the allocation of the prizes depends on the distribution of abilities of all players and their related marginal productivity. For example, even if the contest is extremely hard, it may still be efficient to allocate the entire prize to the most able group member.

We then test the theoretical results in the lab, focusing on the effect of intra-group inequality in contests. Throughout our treatments, we provide a direct comparison

Dependent Variable: BAD			
	Treatment Ability	Treatment Prize	Treatment Combination
Period	-0.217*** (0.0255)	-0.0414* (0.0219)	-0.188*** 0.0255
L-Type	-7.815*** (2.984)	-4.148 (2.686)	-8.488*** (2.955)
H-Type	3.103 (3.126)	1.959 (2.573)	3.362 (2.779)
Competitive Score	0.074 (0.311)	0.249 (0.307)	0.00652 (0.280)
Risk-seeeking	1.526** (0.69)	0.638 (0.753)	1.645** (0.682)
IQ score	0.327 (0.795)	-0.339 (0.771)	0.0195 (0.745)
Female	-0.353 (2.715)	-0.191 (2.203)	-0.572 (2.372)
Constant	10.17 (8.768)	14.71 (8.464)	8.625 (8.732)
Observations	1560	1800	1320

Table 2.5: BAD multi-level mixed effects regression

Note: Multilevel mixed effects models using random intercepts for experimental sessions and individual subjects. L-Type and H-Type are categorical variables standing for the low ability/reward subjects and high ability/reward subjects respectively. The omitted category is the medium type. Female is a dummy variable with female subjects being 1. Risk_seeking ranges from 1 to 10 with higher values indicating more risk-seeking attitudes. IQ score is the total No. of correct answers from 12 questions. Numbers in parentheses indicate standard errors. Significance levels * $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

of the efforts exerted by equal and unequal groups. In contrast to the theory, our experimental result shows that inequality in ability and inequality in rewards do not increase group efficiency. Both of them must be used together to substantially increase a group's probability of winning.

We extend our analysis to understand how players choose their contributions. We find that subjects positively respond to their belief about opponents' contribution, but they there is not a significant relationship between an individual contributions

and her belief about the teammate's decisions.

To conclude we investigate the possible factors behind a player's deviations from the model's prediction. We do this by separating strategic concerns and personal characteristics through a novel definition of BAD. As a result, we show that players on average hold unbiased beliefs, but their contributions are significantly higher than the theoretical predictions adjusted for their beliefs. Further, the BAD in our experiment is significantly correlated with risk attitudes, with risk-seeking subjects having a greater BAD. Finally, BAD diminishes over time. The overall analysis on BAD suggests that error, along with risk-attitudes, may explain overbidding in contest experiments.

Importantly, in our experiment we implemented a mild level of inequality in rewards and found a positive effect on group competitiveness. An interesting question is to construct an empirical calibration of the relationship between the level of inequality and the effectiveness. Although the model predicts a monotonic positive relationship, behavioural factors such as social comparison concerns may make extreme levels of within group inequality harmful for competitiveness. We leave it for future research.

Appendix A

Lemma 2.2. *Let $\phi_i^o(\phi_{ik}^o, \dots, \phi_{in_i}^o)$ be the prize allocation that maximises group i 's probability of winning at the aggregate effort X . If ϕ_i^o is the solution of $\phi_i \in \operatorname{argmax} \sigma_i = s_i(X, \phi_i)$, then it is also the solution of $\phi_i \in \operatorname{argmax} X_i$. Formally,*

$$\phi_i \in \operatorname{argmax} \sigma_i \Leftrightarrow \phi_i \in \operatorname{argmax} X_i$$

Proof. Note that $\sigma_i = X_i/X$. Hence, at a fixed X , it clearly holds that $\phi_i \in$

$$\operatorname{argmax} \sigma_i \Leftrightarrow \phi_i \in \operatorname{argmax} X_i.$$

□

Proof of Proposition 2.2.1

A Subgame Perfect Nash Equilibrium in pure strategies is a profile of prize allocations $(\phi_i^*, \dots, \phi_N^*)$ and aggregate effort X^* that simultaneously solve the following equations:

$$\begin{aligned} & \phi_i \in \operatorname{argmax} \sigma_i \quad \forall i \\ \text{s.t. } & \sum_k^{n_i} \phi_{ik} = \Phi_i, \quad \phi_{ik} \geq 0 \quad \forall k; \end{aligned} \tag{2.6}$$

$$\sum_i^N s_i(X, \phi_i) = 1; \tag{2.7}$$

where $\sigma_i = s_i(X, \phi_i)$, as defined in the proof of Lemma (2.1), and implicitly by

$$\sigma_i = \sum_{k=1}^{n_i} f\left(\frac{1 - \sigma_i}{X}(P_i + \phi_{ik})v_{ik}\right) / X. \tag{2.8}$$

In Lemma 2.1 we proved that for any profile of prize allocations (ϕ_1, \dots, ϕ_N) there exists a unique equilibrium aggregate $0 < X^* < \infty$ such that $\sum_i^N \sigma_i^* = 1$ and $0 < \sigma_i^* < 1 \quad \forall i$, where $\sigma_i^* = s_i(X^*, \phi_i)$. Hence, it must hold that at an equilibrium profile of prize allocations $(\phi_1^*, \dots, \phi_N^*)$ the aggregate X^* is unique, i.e. we proved existence. On the other hand, we have to prove that there is only one $0 < X^* < \infty$ that simultaneously solves for Equation (2.6-2.7).

Now, note that for any $0 < X < \infty$, the left-hand side of (2.8) exceeds the right at $\sigma_i = 1$, while the right-hand side exceeds the left at $\sigma_i = 1$ and it is decreasing on σ_i . It implies that there is a unique $0 < \sigma_i < 1$ that solves (2.8) for any $0 < X < \infty$ and

prize allocation $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$. Thus, for a fixed aggregate $0 < X < \infty$, the group i can achieve any probability of winning $\sigma_i \in [\sigma_i^L, \sigma_i^H]$ by choosing the appropriate $\phi_i = (\phi_{i1}, \dots, \phi_{in_i})$. Clearly, there may be more than one ϕ_i that achieves the same $\sigma_i \in [\sigma_i^L, \sigma_i^H]$. In order to find σ_i^H , which is the highest probability of winning that group i can reach at a given aggregate $0 < X < \infty$, we could either take the implicit derivative of function (2.8) or solving the following equivalent system of equations:

$$\begin{aligned} \phi_i \in \operatorname{argmax} \quad & \sum_{k=1}^{n_i} f\left(\frac{1-\sigma_i}{X}(P_i + \phi_{ik})v_{ik}\right) / X \quad \forall i \\ \text{s.t.} \quad & \sum_k^{n_i} \phi_{ik} = \Phi_i, \quad \phi_{ik} \geq 0 \quad \forall k, \end{aligned} \quad (2.9)$$

$$\sigma_i = \sum_{k=1}^{n_i} f\left(\frac{1-\sigma_i}{X}(P_i + \phi_{ik}^o)v_{ik}\right) / X \quad (2.10)$$

where ϕ_{ik}^o is the solution of (2.9). In other words, we maximise the right-hand side of (2.8) under the condition that $\sigma_i^H = s_i(X, \phi_i^o)$.

Easy contests

In easy contests, the right-hand side of Equation (2.8) is a sum of strictly convex functions, i.e. it is strictly convex. Thus, for any $0 < \sigma_i^H < 1$, the solution of (2.9) is a corner solution. It implies that there are n_i^h equilibrium allocation that maximises the group i 's probability of winning, where n_i^h is the number of players with the highest ability in group i . Thus, we are left with proving that there is a unique $0 < X^* < \infty$ that solves for $\sum_i^N \sigma_i^H = 1$, i.e. $\sum_i^N s_i(X^*, \phi_i^o) = 1$. However, this follows immediately from Lemma 2.1.

Hard contests

In hard contests, the right-hand side of Equation (2.8) is a sum of strictly concave functions, i.e. it is strictly concave. Hence, the solution of (2.9) is interior for any $0 < X < \infty$ and $0 < \sigma_i < 1$ and has to satisfy

$$\frac{\frac{1-\sigma_i}{X}v_{ik}}{g''(x_{ik})} - \lambda = 0 \quad \forall k. \quad (2.11)$$

Let, without loss of generality, $v_{i1} \geq \dots \geq v_{in_i}$. From the system of equations (2.11) we have that

$$\frac{v_{i1}}{g''(x_{i1})} \geq \frac{v_{ik}}{g''(x_{ik})} \quad \forall k > 1. \quad (2.12)$$

Now, recall that the left-hand side of (2.10) exceeds the right-hand side at $\sigma_i^H = 1$, while the right-hand side exceeds the left at zero. Furthermore, (2.10) is decreasing on σ_i , indeed using (2.12) we have that

$$\sum_{k=1}^{n_i} \frac{-v_{ik}(\phi_{ik} + P_i)}{g''(x_{ik})} < -\sum_{k=1}^{n_i} \frac{v_{ik}(1 - \sigma_i)}{g''(x_{ik})} \frac{\partial \phi_{ik}^o}{\partial \sigma_i} = 0.$$

Thus, there is only one pair of $0 < \sigma_i^H < 1$ and ϕ_i^o that simultaneously solves for (2.9-2.10) for any $0 < X < \infty$. Finally, we are left with proving that there is a unique $0 < X^* < \infty$ that solve for $\sum_i^N \sigma_i^H = 1$, i.e. $\sum_i^N s_i(X^*, \phi_i^o) = 1$. However, because $\sigma_i = s_i(X, \phi_i^o)$ is strictly decreasing and continuous in X for all i , $\lim_{X \rightarrow \infty} s_i(X, \phi_i^o) = 0$, and $\lim_{X \rightarrow 0} s_i(X, \phi_i^o) = 1$, then by the intermediate value theorem there is a unique value of X^* such that $\sum_i^N s_i(X^*, \phi_i^o) = 1$. Clearly, at this X^* the equilibrium allocation $\phi_i^o = \phi_i^* = (\phi_{i1}^*, \dots, \phi_{in_i}^*) \forall i$.

part ii)

Note that Equation (2.12) always holds with strict equality if two members have the same ability. However, as shown in part (iii) (see below) it is still possible that two or more players with different abilities do not receive a positive share of the prize.

part iii)

Suppose $g''(x_{ik}) > 0$ at $\phi_{ik} = 0$. This can occur for two reasons: the public part of the prize $P_i > 0$ implies $x_{ik} > 0 \forall k$, and $g''(0) > 0$. Thus, we might have that

$$\frac{v_{i1}}{g''\left(\frac{1-\sigma_i^*}{X^*}(P_i + \Phi_i)v_{i1}\right)} > \frac{v_{im}}{g''\left(\frac{1-\sigma_i^*}{X^*}P_i v_{im}\right)} \quad \forall m > 1. \quad (2.13)$$

Appendix B

The following instructions are for Treatment Ability and have been read out loud by the experimenter.

Experimental instructions

You are about to participate in an experiment in the economics of decision-making. These instructions are meant to clarify how the experiment actually works and how you earn money in the experiment. Your earnings will be paid to you **privately in cash** at the end of the experiment. To ensure the best results for yourself, and accurate data for the experimenters, please **do not communicate** with the other participants at any point during the experiment. If you have any questions, or need assistance of any kind, raise your hand and an experimenter will come to you. Economics experiments have a strict policy against deception. If we do anything deceptive, or don't pay you in cash as described, then you can complain to the school of Economics at the University of Edinburgh and we will be in serious trouble. The

currency in this experiment is expressed in points. Your points will be converted to cash and paid to you at the end of the experiment privately, based on the exchange rate.

The currency in this experiment are expressed in points. Your points will be converted to cash and paid to you at the end of the experiment privately, based on the exchange rate.

1500 points = \$4.

In addition you will be paid \$3 for participation and a bonus (\$2 + a lottery) for completing all survey questions at the end of this experiment.

The experiment

This experiment involves a decision-making task in groups. The same task will be played a total of 30 times (rounds). You will not know who your group members are neither during nor after the experiment. You will be randomly rematched into a new group after each round. At the beginning of the experiment, you will be randomly assigned to one of the three types - X, Y or Z. Types will remain **fixed** until the end of the entire experiment.

Groups and matching

First, before each round, all participants will be randomly divided into groups of two assigned in the following way:

XX groups and YZ groups.

If your type is X, then you will always be in a group with another X type player.

If your type is Y, then you will always be in a group with a Z type player.

If your type is Z, then you will always be in a group with a Y type player.

Second, your group will randomly match with a group of the other type. Hence, if you are in a XX group, then you are always matched with a YZ group and vice versa. Finally, after each round all groups are dissolved, all participants will be randomly assigned (again) into groups according to their types, and then the groups will randomly re-match.

The task

For every round, your group is competing against your matched group for a reward worth 1000 points. If your group wins, the reward will be divided equally between the two of you.

All participants begin each round with an endowment of 300 points and choose a contribution to the group account. The minimum No. of contributions is 0 and the maximum is 50, and any integer between 0 and 50 is also allowed. The group account is the sum of the contributions of its members. Contributions have a cost based on the participants' type and details are listed on Table 1 (*a separate piece of paper on your desk*). You are allowed to contribute with costs higher than your endowment, by paying more points than your endowment. Doing so may result in a negative payoff for that specific round, however, at the end of the experiment, we will make sure you earn at least the show-up fee.

The chance that your group receives the reward in a round depends on the contributions on your group account and your matched group's account. At the start of each round, all 4 participants (you, your group member, and the two participants in the other group) will decide how much to contribute simultaneously. Once the

contribution decisions are made, a computerized lottery will determine which group will receive the reward.

In this lottery draw there are 2 types of tickets: type XX and type YZ. Each type of ticket corresponds to the group who will receive the reward if a ticket of this type is drawn. Thus, if a type XX ticket is drawn, then group XX wins. If a type YZ ticket is drawn, then group YZ wins. The reward will be equally shared between the winning group members.

The number of tickets of each type corresponds exactly to the contributions on the group account.

No. of XX tickets = No. of contributions by member X + No. of contributions by member X.
No. of YZ tickets = No. of contributions by member Y + No. of contributions by member Z.

Every ticket is equally likely to be drawn by the computer.

In addition to the above task, while you are deciding your contribution, you will be asked to predict (1) the total contribution on your group's and (2) on the other group's account. For every correct prediction, you will receive 50 points (0 for incorrect predictions).

An example

Suppose the contributions on group XX's account are 32 (13 + 19), and the contribution on group YZ's account is 15 (5 for Y + 10 for Z). There will be a total of 47 (32+15) tickets and each ticket is equally likely to be the winning one. The feedback will be shown to you as following:

In this example, the winning group, XX players have a payoff calculated as:

group	Contributions	Tickets	Chance of winning	Winner ticket	Winning group
XX	32	1-32	0.68	8	XX
ZY	15	33-47	0.32	8	XX

$$Payoff = reward/2 + endowment - individual\ cost\ of\ contribution + prediction\ profit.$$

Then, in points is $Payoff = 1000/2 + 300 - \text{individual cost of contribution} + 50 \cdot \text{No. of correct prediction}$.

On the other hand, the other group, YZ players have a payoff calculated as:

$$Payoff = endowment - individual\ cost\ of\ contribution + prediction\ profit.$$

Note that each player has her own contribution decision and the corresponding costs (listed in Table 1). Numbers in the example are for illustrative purpose and in no way they suggest what you should do in the actual experiment.

At the end of the experiment, 5 rounds will be randomly selected for actual payments and you will earn the sum profit of these 5 rounds. Thus it is in your best interest to make serious decisions for every round.

Feedback

At the end of each round, you will receive feedback information on your group's contributions, the other group's contributions, the winning group and your profit in this round.

Practice questions

Before the start of today's session, please answer the practice questions shown on your screen. Feel free to go back and check the instructions while answering these questions.

Trial round

After the practice questions, you will experience a trial round which will not be selected for payment. After the trial round, you will be given additional opportunities to ask questions. After which, the 30 rounds eligible for payments begin.

The survey

After the end of round 30, you will be asked to participate in a survey. Instructions for completing the survey will be shown on your screen. At the end the survey, a bonus reward will be provided.

The end of the experiment

Please remain seated and follow the instructions by leaving the room one by one to receive your payment. Thank you very much.

The cost table (for Treatment Ability)

This table specifies the cost of contribution for different types. For example, if you are type X, by choosing to contribute 7 tickets, it costs you 52 points.

No. of contributions	X type	Y type	Z type
0	0	0	0
1	5	10	3
2	11	23	8
3	19	37	12
4	26	53	18
5	34	69	23
6	43	86	29
7	52	103	34
8	61	121	40
9	70	140	47
10	79	158	53
11	89	178	59
12	99	197	66
13	109	217	72
14	119	237	79
15	129	258	86
16	139	279	93
17	150	300	100
18	160	321	107
19	171	342	114
20	182	364	121
21	193	386	129
22	204	408	136
23	215	431	144
24	227	453	151
25	238	476	159
26	249	499	166
27	261	522	174
28	273	545	182
29	284	569	190
30	296	592	197
31	308	616	205
32	320	640	213
33	332	664	221
34	344	688	229
35	356	713	238
36	369	737	246
37	381	762	254
38	393	787	262
39	406	811	270
40	418	837	279
41	431	862	287
42	443	887	296
43	456	912	304
44	469	938	313
45	482	964	321
46	495	989	330
47	508	1015	338
48	521	1041	347
49	534	1067	356
50	547	1093	364

Choose your contribution and guess other teams contribution. Please use the cost table that has been provided to you to make your decisions!

Group

MMI

Type

M

Your endowment

300

Your contribution

Guess your team total contribution

Guess the other team total contribution

OK

Figure 2.4: Contribution stage

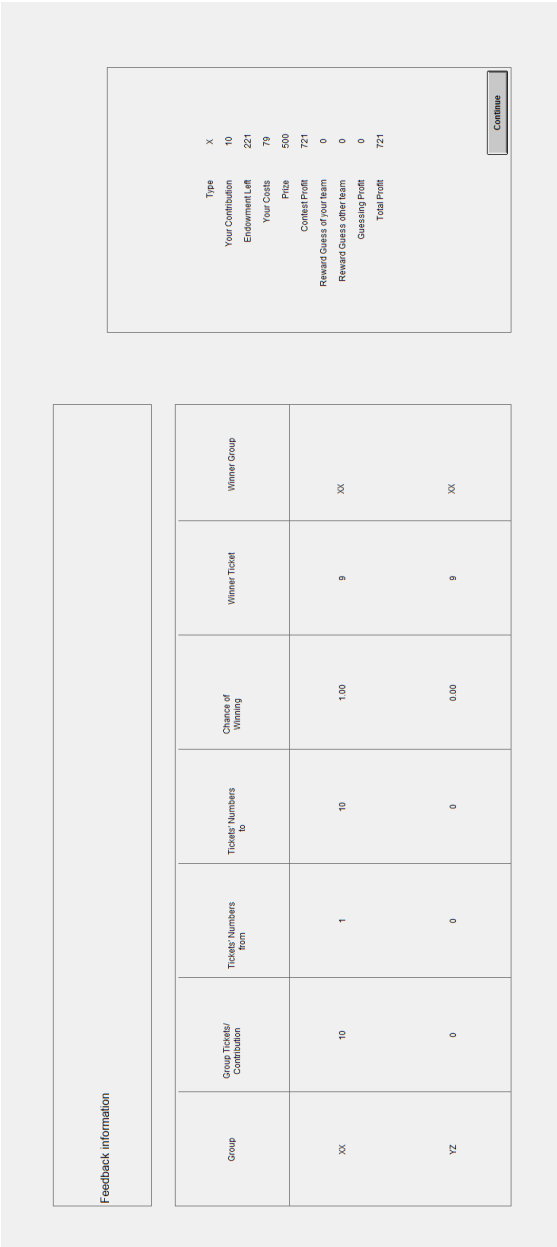


Figure 2.5: Feedback

Appendix C. Additional material.

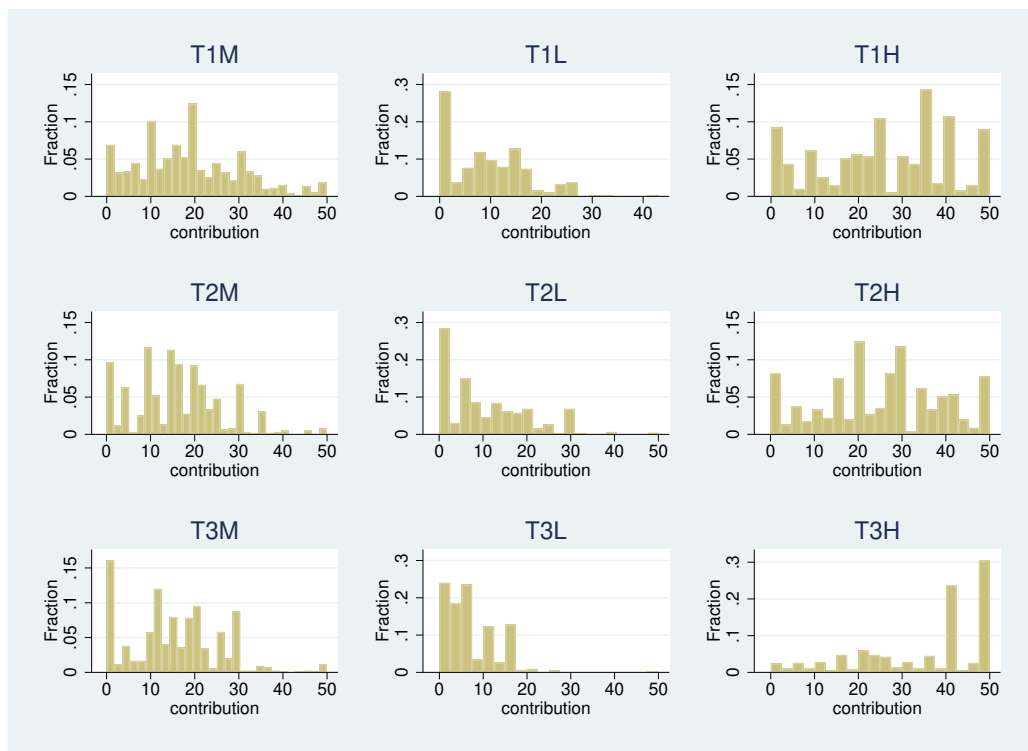


Figure 2.6: Distribution of contribution across treatments and types

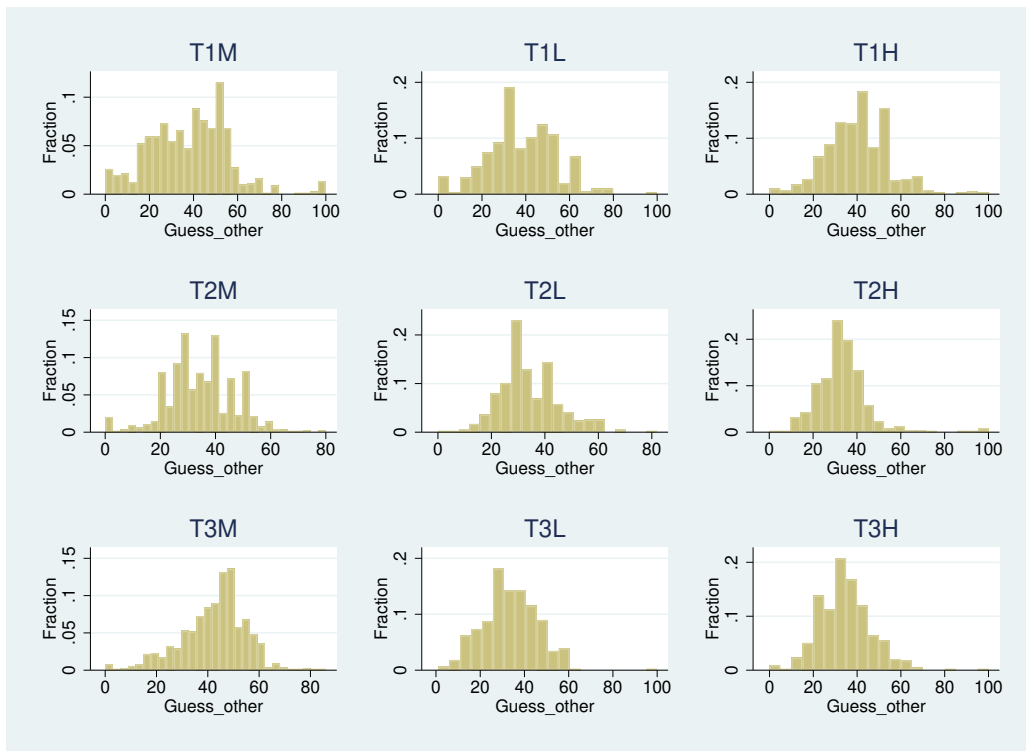


Figure 2.7: Belief about the other group across treatments and types

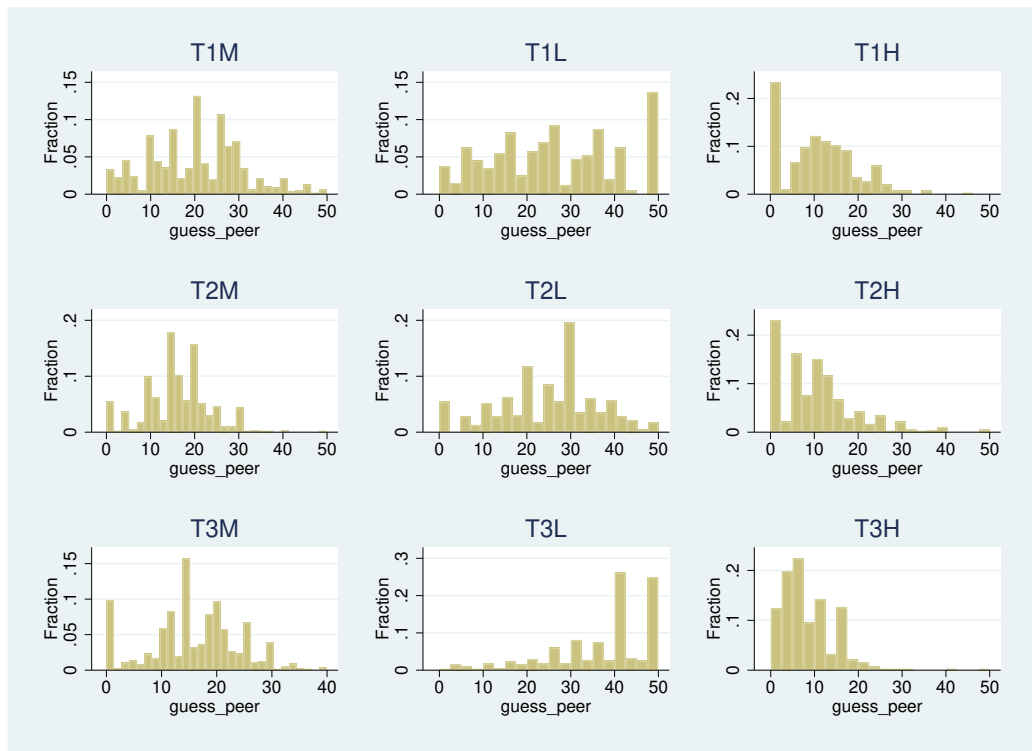


Figure 2.8: Belief about the peer member across treatments and types

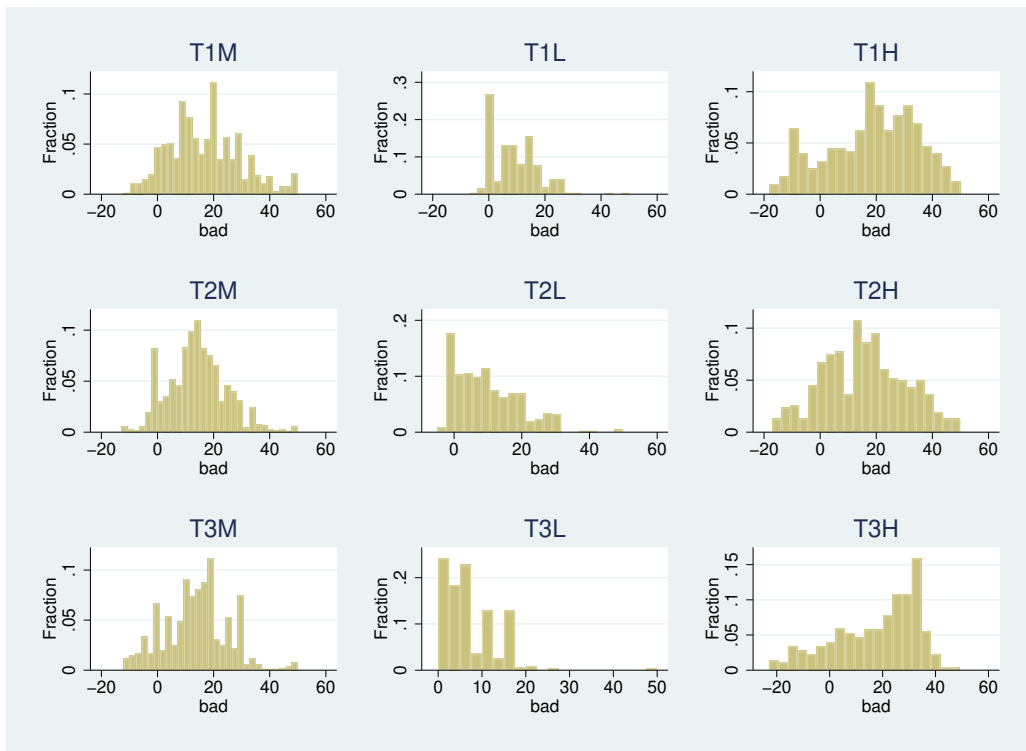


Figure 2.9: BAD across treatments and types

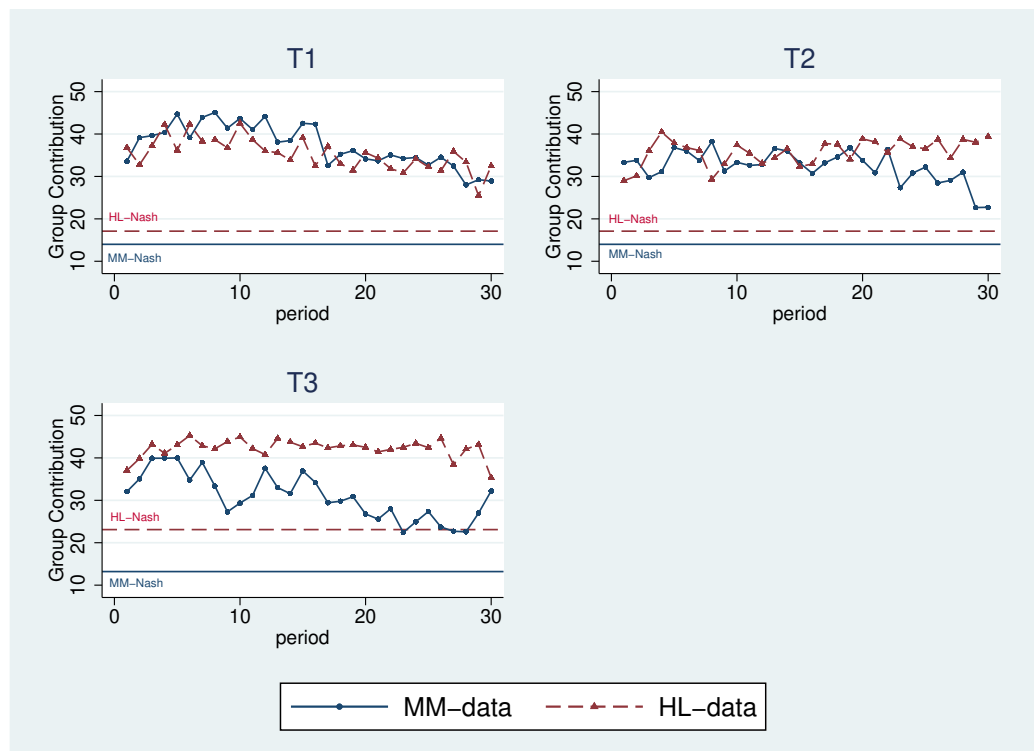


Figure 2.10: Teams' contributions over time

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Chapter 3

War of Attrition with Outside Options

3.1 Introduction

Models of auctions have increasingly attracted attention for their empirical and theoretical importance. Today, a colossal amount of resources is allocated through auctions. Goods and services are sold in online auction platforms and procurement offers, as well as firms' privatisation, are made through auctions. Furthermore, all-pay auctions, together with contests in the style of Tullock (1967), are established tools for modelling situations in which every bid pays regardless of the outcome. Examples for this type of auctions include students competing for grades on a curve, workers competing for jobs, firms contending the monopoly of a market, and countries fighting over a territory.

In many of the situations above, bidders have possibly diverse valuations of the prize, but also diverse outside options. For instance, buyers have differing willingness to pay because they privately know of an alternative item they can buy to substitute the one auctioned. Another example includes the non-refundable investments made by two firms competing in a R&D race. The volume of the investments depends on

the profits the firms would make winning the patent, but also on the profits they would make if the patent is won by someone else.

In this paper, we focus on the war of attrition (second price all-pay auctions) with outside options and incomplete information. We propose a model with the following features. Players receive a pair of signals: their valuation of the prize, and their valuation of the outside option. Signals within the same pair can be correlated whereas signals between pairs are independent. In other words, if a player has a high valuation of the prize when signals are (positively) correlated, then she probably has a high valuation for the outside option. Nonetheless, this does not imply that another player has a greater probability of having higher signals as well.

Using this framework, we show that our model can be reduced to a standard war of attrition in one signal. The value of the “new” signal equals the difference between the pair of valuations received by a player, and its distribution can be derived using the distributions of the valuations the prize and of the outside option. We then describe the symmetric equilibrium bidding strategy and the expected payments of the game.

In the model, we allow for the possibility that a player may prefer the outside option over the prize. As a result, it may occur that only a subset of the players takes part in the competition. We call this set of players “participants”.

Under the assumption that the participants know the exact number of competitors they face before choosing their bid, we show that the expected aggregate of the payments of the auction is the weighted average of the expected aggregate payments for any number of participants. The weights are given by a binomial distribution, where the probability of success corresponds to the probability that a player’s valuation of the prize is higher than her valuation of the outside option.

We also discuss the case in which all players take part in the auction, even those

with a higher valuation of the outside option. Despite these players bidding zero, their participation in the contest substantially changes the strategy of the players interested in winning the prize. To see why this occurs let us focus on the information available to a player with a higher valuation of the prize. Contrary to the previous case, she does not know how many players will bid a strictly positive amount. What she knows instead is the probability that a random player makes a positive bid, as it is equal to the probability that a player's valuation of the prize is higher than the valuation of the outside option. Using this information, a player derives the probability of competing against every number of strictly positive bids. The probability mass function of every event can be calculated using the binomial distribution since pairs of signals are independent draws. Finally, players have to choose one bidding strategy for all possible events. We show that the equilibrium strategy is a weighted average of the strategies they would adopt knowing the number of strictly positive bids.

We conclude by undertaking comparative static for a specific case of distribution functions. Our analysis shows that a more spread out distribution of the valuations increases the expected aggregate of the payments.

Literature Review

The first game theoretic approach to auctions can be found in Vickrey (1961), who also made a considerable contribution on the well-known revenue equivalence theorem. As discussed in Myerson (1981), when the bidders are risk-neutral and values are independently and identically distributed, the “revenue equivalence principle” states that many types of auction formats have the same expected revenue for the seller. On the other hand, if the players' valuations are affiliated, then the war of attrition raises higher expected revenue than the other auction structures (see Milgrom

and Weber (1982) and Krishna and Morgan (1997)).

Other work on optimal auctions are by Jehiel et al. (1996, 1999). In situations where the sale in an auction affects the future interactions between potential buyers, the authors use a mechanism design approach to construct the revenue-maximizing auction for the seller. Their model assumes that agents who do not acquire the object are characterised by an identity-dependent externality. In a complete information model with identity-dependent externalities, Klose and Kovenock (2015) derive a necessary and sufficient condition for the existence of equilibria in the all-pay auction. Contrary to our model, the outside options in these papers are endogenously determined either by the mechanism itself or by others' strategies.

McAfee and McMillan (1987) and Matthews (1987) study first-price auctions with a stochastic number of risk-averse bidders. While they do not characterise the equilibrium strategies, they study whether it is better to reveal or conceal the information about the number of bidders. In a setting with independent private signals, Harstad et al. (1990) show that equilibrium strategies in first and second price auctions with a stochastic number of bidders are weighted averages of the equilibrium strategies when the number of bidders is common knowledge. Recently, Bos (2012), shows that the same result holds in a model of war of attrition.

A framework complementary to ours is examined in Green and Laffont (1984), where the authors study auctions with reservation utilities. In their model, players decide whether to take part in the competition by comparing the expected utility of doing so to their own reservation utility. If a player takes part in an auction, then she gives up her reservation utility. In our model, on the other hand, only the winner has to renounce to her outside option.

A model of war of attrition with outside option can be found in Hafer (2006). In the first part of the paper, the author studies conflicts with incomplete information in

which players receive two signals and proves the equilibrium existence. Although this similarity, the way players' payoff are defined substantially differs from our model. Further, the author assumes that the two signals are independent, identically and uniformly distributed.

To the best of our knowledge, the closest paper to ours is Kirchkamp et al. (2009). The authors extend the first price and second price auction with independent private values to allow for public and private outside options. They derive the symmetric equilibrium strategies of the game and test the prediction in the laboratory. There are three main differences between their theoretical model and ours. First, we study war of attrition models while they focus on first and second price auction. Second, they assume that all players strictly prefer to compete as they always value the prize more than the outside option. Finally, we undertake comparative statics on the distribution of the signals.

The paper is organised as follows. Section 3.2 surveys the main results of the standard war of attrition; Section 3.3 presents the model with outside option and related applications; Section 3.4 concludes.

3.2 The standard war of attrition

Readers who are familiar with the model of war of attrition (or second-price all-pay auction) can skip this section.

There are n risk-neutral players that compete for a prize. Prior to the contest each agent i receives a private signal, v_i , that affects her valuation of the prize. Individuals are symmetric, and each of them knows their own valuation and perceives the other $n - 1$ valuations to be random draws from the distribution $F(v)$ on $[\underline{v}, \bar{v}]$, where $\underline{v} \geq 0$. Altogether, the payoff of individual i is

$$\pi_i = \begin{cases} v_i - b_i, & \text{if } b_i > \max_{j \neq i} b_j \\ -b_i, & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Let $\Pr(b_i)$ be the probability that $b_i > \max_{j \neq i} b_j$, i.e. that i is the player with the highest bid, and let $p(b_i)$ be the probability that $b_i = \max_{j \neq i} b_j$, then the player i 's expected payoff is

$$\Pi_i(b_i) = (1 - \Pr(b_i))(-b_i) + \int_0^{b_i} p(t)(v_i - t)dt. \quad (3.1)$$

Player i has to choose the highest bid b_i she is willing to commit in order to attempt to win the prize and then have a payoff of $(v_i - b_i)$. Suppose that agent $j \neq i$ follows the strategy $\beta(v_j)$, where β is differentiable and increasing. Since β is increasing, then $\Pr(b_i > \max_{j \neq i} b_j) = F^{n-1}(v_i)$ and bidder i 's expected payoff in (3.1) can be rewritten as

$$\Pi_i(b_i) = (1 - F^{n-1}(\beta^{-1}(b_i)))(-b_i) + \int_{\underline{v}}^{\beta^{-1}(b_i)} (t - \beta(t))dF^{n-1}(t). \quad (3.2)$$

The first-order condition is

$$\frac{\partial \beta^{-1}(b_i)}{\partial b_i} = \frac{1 - F^{n-1}(\beta^{-1}(b_i))}{v_i(n-1)f(\beta^{-1}(b_i))F^{n-2}(\beta^{-1}(b_i))}.$$

Given that in the symmetric equilibrium $\beta^{-1}(b_i) = v_i$, it must also be that $\frac{\partial \beta^{-1}(b_i)}{\partial b_i} = 1/(\frac{\partial \beta(v_i)}{\partial v_i})$, which yields

$$\frac{\partial \beta(v_i)}{\partial v_i} = \frac{v_i(n-1)f(v_i)F^{n-2}(v_i)}{1 - F^{n-1}(v_i)}dt = v_i h(v_i), \quad (3.3)$$

where $h(v_i)$ is the hazard rate of $F^{n-1}(v_i)$. Finally,

$$\beta(v_i) = \int_{\underline{v}}^{v_i} th(t)dt.$$

The derivation of β is only heuristic as its optimality when all $j \neq i$ adopt $\beta(v_j)$ has not been established. The next lemma provides sufficient conditions for β to be a symmetric equilibrium.

Lemma 3.1. *Suppose that, for all v , $vh(v)$ is an increasing function. The equilibrium strategy of the standard war of attrition is defined by*

$$\beta_{WOA}(v) = \int_{\underline{v}}^v th(t)dt. \quad (3.4)$$

Proof. Suppose that all $j \neq i$ follow the strategy $\beta(v_j)$, while i plays as if her signal was w_i , then the optimal w_i has to satisfy

$$\Pi(w_i) = (1 - F^{n-1}(w_i))(-\beta(w_i)) + \int_{\underline{v}}^{w_i} (v_i - \beta(t))dF^{n-1}(t).$$

The first-order condition with respect to w_i is

$$\beta'(w_i) = v_i h(w_i)$$

By substituting $\beta'(w_i)$ with Equation (3.3) we have $w_i = k_i$. □

The following lemma describes the expected payments of the game.

Lemma 3.2. *In the standard war of attrition, the expected payment for a player who receives signal v is defined by*

$$\hat{b}_{WOA} = \int_{\underline{v}}^v t dF^{n-1}(t). \quad (3.5)$$

The expected aggregate of all the payments is

$$\hat{B}_{WOA} = N \int_{\underline{v}}^{\bar{v}} \hat{b}_{WOA} dF(t) = N \int_{\underline{v}}^{\bar{v}} dF(t) \int_{\underline{v}}^v t dF^{n-1}(t). \quad (3.6)$$

Proof. The expected payment of player i with signal v_i is

$$\begin{aligned}\hat{b}_i &= (1 - F^{n-1}(v_i))\beta(v_i) + \int_{\underline{v}}^{v_i} \beta(t) dF^{n-1}(t) \\ &= \int_{\underline{v}}^{v_i} \beta'(t) dt - \int_{\underline{v}}^{v_i} F^{n-1}(t) \beta'(t) dt \\ &= \int_{\underline{v}}^{v_i} (1 - F^{n-1}(t)) \beta'(t) dt.\end{aligned}$$

Substituting $\beta'(t)$ with (3.3) completes the proof. \square

In the next section, we introduce a war of attrition where players have two private signals: the valuation of the prize and the valuation of the outside option. As we will see, this game can be reduced to the standard war of attrition.

3.3 The war of attrition with outside options

Typically, bidders who compete in auctions do not only have different valuations for the prize, but also different outside options. Consider, for example, two companies investing to get the monopoly of a market.¹ Their decision on how much to invest must depend on the profit they can make, but also on the profit they would make in a different market. Alternatively, one can interpret the outside option as the profit a company would make if a competitor becomes the monopolist. Finally, it can also represent an individual's payoff at the time in which the contest takes place. On the other hand, the valuation of the prize, represents the payoff level a player reaches by winning the competition.

As we will see below, the model is general enough to include cases in which players have negative outside options (negative starting payoffs) and negative prize valua-

¹A recent case on this is the competition for cloud services between Amazon, Google and Microsoft, see <https://www.economist.com/business/2014/08/30/silver-lining>.

tion, but they still participate in the auction as winning would improve their status. Clearly, in all the examples above, winning the auction automatically excludes the use of the outside option, however, losing does not imply players have to renounce it.

3.3.1 Preliminaries

There are n risk-neutral players who compete for a prize. Prior to the contest player i receives two private signals: v_i represents her valuation for the prize, while o_i her valuation for the outside option. Players are symmetric, each of them knows her own type (v_i, o_i) and perceives that the valuations of others are random draws from the distributions $F(v)$ on $[\underline{v}, \bar{v}]$ and $G(o)$ on $[\underline{o}, \bar{o}]$, respectively. Note that $F(v)$ and $G(o)$ do not need to be cdf of the same family, do not need to be independent, and there are no assumptions on their support. Further, we assume that winning the contest automatically cancels out the winner's outside option. In other words, among all players it is only the winner who enjoys the prize and renounces her outside option. Altogether, the payoff of individual i is

$$\pi_i = \begin{cases} v_i - \max_{j \neq i} b_j, & \text{if } b_i > \max_{j \neq i} b_j \\ o_i - b_i, & \text{if } b_i < \max_{j \neq i} b_j \end{cases}$$

Player i chooses the bid b_i that maximises her payoff. Let $\Pr(b_i)$ be the probability that $b_i > \max_{j \neq i} b_j$, i.e. player i is the player with the highest bid, and let $p(b_i)$ be the expected probability that $b_i = \max_{j \neq i} b_j$, then player i expected payoff is

$$\Pi_i(b_i) = (1 - \Pr(b_i))(o_i - b_i) + \int_0^{b_i} p(t)(v_i - t)dt. \quad (3.7)$$

The first-order condition is

$$\frac{p(b_i)}{1 - \Pr(b_i)}(v_i - o_i) = 1.$$

Note that a symmetric strategy as a function of the two signals cannot be solved because of the infinite (v_i, o_i) that relates to a given b_i . However, it is easy to see that players with the same $z_i = v_i - o_i \geq 0$ must choose the same b_i . Thus, an equivalent problem to the maximization of (3.7) is the maximization of

$$\Pi_i(b_i) = (1 - \Pr(b_i))(-b_i) + \int_0^{b_i} p(t)(z_i - t)dt. \quad (3.8)$$

An alternative way to show the two problems are equivalent is the following:

$$\begin{aligned} \Pi(b_i) &= (1 - \Pr(b_i))(o_i - b_i) + \int_0^{b_i} p(t)(v_i - t)dt \\ &= (1 - \Pr(b_i))(-b_i) - \int_0^{b_i} p(t)o_i dt + \int_0^{b_i} p(t)(v_i - t)dt \\ &= (1 - \Pr(b_i))(-b_i) + \int_0^{b_i} p(t)(v_i - o_i - t)dt. \end{aligned}$$

The game with two signals has been reduced to a game with one signal, $z = v - o$, which equals the difference between the prize valuation and the outside option. The new signal z is distributed accordingly to $\tilde{F}(z)$ with support $[\underline{z}, \bar{z}] = [\underline{v} - \bar{o}, \bar{v} - \underline{o}]$. The explicit distribution of $\tilde{F}(z)$ can be derived using the fact that v and o are distributed accordingly to $F(v)$ and $G(o)$, respectively.

Before moving to the equilibrium analysis of the game, we need to define the behaviour of the players with $z < 0$ as we didn't put any restriction on the support of the distribution functions. There are two cases to be considered. First, these players do not take part in the contest as they already know they strictly prefer the outside option. This, for example, occurs when players know their valuation of the prize before the auction takes place. Second, they take part in the contest and bid zero. This situation accounts for the cases in which players receive their valuation

for the prize only if they participate in the competition. While for the former case we characterise the equilibrium strategies and expected payments, for the latter we limit our analysis to the existence of symmetric strategies.

3.3.2 Case 1

Assume that n players receive their signals, but those with $z < 0$ do not take part in the competition. It is public knowledge the number of $k \leq n$ players will participate in the contest and bid. We use the word “participant” when we refer to any one of these k players. As for the standard war of attrition, suppose that participant $j \neq i$ follows the strategy $\beta(z_j)$, where β is differentiable and increasing in $z_j = v_j - o_j \geq 0$. Since β is increasing, then participant i ’s probability of having the highest bid equals the probability of having the highest signal among the k participants (who have a signal $z > 0$). Formally, $\Pr(b_i > \max_{j \neq i} b_j) = \Pr(z_i > \max_{j \neq i} z_j) = \ddot{F}^{k-1}(z_i)$, where $\ddot{F}(z_i) = \frac{\dot{F}(z_i) - \dot{F}(0)}{1 - \dot{F}(0)}$. The participant i ’s expected payoff is

$$\Pi_i(b_i) = (1 - \ddot{F}^{k-1}(\beta^{-1}(b_i)))(-b_i) + \int_0^{\beta^{-1}(b_i)} (t - \beta(t)) d\ddot{F}^{k-1}(t). \quad (3.9)$$

It is easy to see that the solution of the maximization problem of (3.9) is equivalent to solution of the standard war of attrition² with the caveat that the new cdf is $\ddot{F}^{k-1}(z_i)$, i.e. the truncated cdf, from 0 to \bar{z} , of $\tilde{F}(z)$.

Proposition 3.3.1. *Let $\ddot{h}(z)$ be the hazard rate of $\ddot{F}^{k-1}(z_i)$. Suppose that, for all $z \in [0, \bar{z}]$, $z\ddot{h}(z)$ is an increasing function. For any $k = (2, \dots, n)$, the equilibrium strategy of the war of attrition with outside option is defined by*

$$\beta_O(z) = \int_0^z \frac{t(k-1)\ddot{f}(t)\ddot{F}^{k-2}(t)}{1 - \ddot{F}^{k-1}(t)} dt = \int_0^z t\ddot{h}(t) dt. \quad (3.10)$$

²See equation (3.8).

Proof. The proof follows directly from Lemma (3.1). \square

It is also possible to recover participant i 's strategy in terms of her signals v and o , using the related distributions. Nonetheless, to state the following results there is no need of specifying strategies as a function of the signals v and o .

Proposition 3.3.2. *The equilibrium strategy $\beta_O(z)$ is increasing in v and decreasing in o . Furthermore, $\lim_{z \rightarrow 0} \beta_O(z) = 0$ $\lim_{z \rightarrow \bar{z}} \beta_O(z) = \infty$.*

Proof. As the equilibrium strategy is increasing in $z = v - o$, then, all things being equal, it is increasing in v and decreasing in o .³ The rest of the proof follows from the two limits: $\lim_{z \rightarrow 0} \int_0^z t \ddot{h}(t) dt = 0$ and $\lim_{z \rightarrow \bar{z}} \int_0^z t \ddot{h}(t) dt = \infty$. The same result holds for $z > 0$ by proposition 1 in Krishna and Morgan (1997). \square

In addition, we can also describe the participants' expected contributions and the related expected aggregate of payments.

Proposition 3.3.3. *For any $k = (2, \dots, n)$, the expected payment of a participant is defined by*

$$\hat{b}_O = \int_0^z t d\ddot{F}^{k-1}(t). \quad (3.11)$$

The expected aggregate of k payments is

$$\hat{B}_O(k) = k \int_0^{\bar{z}} \hat{b}_O d\ddot{F}(t) = k \int_0^{\bar{z}} d\ddot{F}(t) \int_0^z t d\ddot{F}^{k-1}(t), \quad (3.12)$$

Proof. The proof follows directly from Lemma (3.2). \square

Importantly, if the lower bound of the support of $\tilde{F}(z)$ is greater than or equal to zero, then all n players participate in the auction. The probability of having the

³Strictly increasing if $z > 0$.

highest signal reduces to $\ddot{F}^{k-1}(z) = \tilde{F}^{n-1}(z)$, as it is unnecessary to truncate $\tilde{F}(z)$. In this case, the expected aggregate of the payments is $\hat{B}_O(k = n)$. On the other hand, when the lower bound of the support of $\tilde{F}(z)$ is lower than zero, players may receive a negative signal and do not participate in the contest. So, the expected aggregate of an auction with n players is the probability-weighted average of the expected aggregate $\hat{B}_O(k)$ for $k = (0, 1, \dots, n)$.

As already discussed above, the signals are independent random draws, and a player's probability of having a signal $z \geq 0$ is $1 - \tilde{F}(0)$. So, the probability of having a contest with k participants (out of the total number of players n) is the probability of getting exactly k successes in n independent Bernoulli trials, where the probability of success is $1 - \tilde{F}(0)$ and the probability of failure is $\tilde{F}(0)$.

Proposition 3.3.4. *Let $k = (0, 1, \dots, n)$ be the number of participants out of a total of n players and $p_k = \binom{n}{k}(1 - \tilde{F}(0))^k \tilde{F}(0)^{n-k}$ the related probability that event k occurs. The expected aggregate of the payments is*

$$\hat{B}_O(n) = \sum_{k=2}^n p_k \hat{B}_O(k). \quad (3.13)$$

Furthermore, the probability that the aggregate payment is zero is

$$p_0 + p_1. \quad (3.14)$$

3.3.3 Case 2

Suppose that all n players take part in the auction, and those with $z \leq 0$ bid zero. Contrary to the previous section, players do not know how many $k \leq n$ will bid, as there is uncertainty over the number of players interested in winning the prize. In

order words, the k participants with signal $z > 0$ do not know how many (strictly positive) bids they have to beat.

A player i , with $z > 0$, faces a stochastic number of competitors as the probability that a random player $j \neq i$ bids a positive amount is $1 - \tilde{F}(0)$. Thus, the probability that $k = (0, 1, \dots, n)$ bids are positive is $p_k = \binom{n}{k} (1 - \tilde{F}(0))^k \tilde{F}(0)^{n-k}$.

Assume as before that agent $j \neq i$ follows the strategy $\beta(z)$, where β is differentiable and increasing in $z > 0$. The expected payoff of player i can be written as

$$\Pi_i(b_i) = (1 - \sum_{k=0}^n p_k \ddot{F}^{k-1}(\beta^{-1}(b_i)))(-b_i) + \sum_{k=0}^n p_k \int_0^{\beta^{-1}(b_i)} (t - \beta(t)) d\ddot{F}^{k-1}(t). \quad (3.15)$$

The equilibrium existence follows from the following proposition.

Proposition 3.3.5. *Bos (2012). For a player with $z > 0$, the equilibrium strategy in the war of attrition with a stochastic number of bidders is a weighted average of equilibrium strategies where the number of bidders is common knowledge. For a player with $z \leq 0$, the equilibrium strategy is to bid zero.*

The characterisation of the bidding strategies of the standard auctions with stochastic competition directly applies to our model with two signals, as the symmetric equilibrium strategy of this game is a weighted average of $\beta_O(z)$ defined in (3.10). However, given that it is extremely difficult to specify the weights of the equilibrium strategies, see Example 3 in Bos (2012), we are not able to describe the expected aggregate of the payments.

Applications

A natural question to ask, using the model presented in Case 1, is whether an increase in dispersion of the signals, in the sense of a mean preserving spread of

the distribution, affects the aggregate of the payments. While until now we did not specify the distribution of z , to undertake this exercise we restrict our attention to some specific cases.

Recall that we define $\tilde{F}(z)$ and $\tilde{f}(z)$ as the cdf and pdf of signal $z = v - o$, where v and o are distributed according to $f(v)$ on $[\underline{v}, \bar{v}]$ and $g(o)$ on $[\underline{o}, \bar{o}]$, respectively. Clearly, any change of $\tilde{f}(z)$ is due to a change in $f(v)$, $g(o)$ or both.⁴

Now, let $\tilde{f}'(z)$ be the mean preserving spread of $\tilde{f}(z)$. We ask whether the expected aggregate of the payments, defined by (3.13), is higher when valuations are distributed according to $\tilde{f}'(z)$ or $\tilde{f}(z)$. In other words, we are interested in the change of the expected aggregate of the payments when there is an increase in the uncertainty of the signal z .

In what follows, we consider the mean preserving spread of density functions symmetrically distributed around zero. This assumption substantially facilitates our analysis as the probability that a player bid zero is not affected by a higher dispersion of the valuations, see figures (3.1) and (3.2).

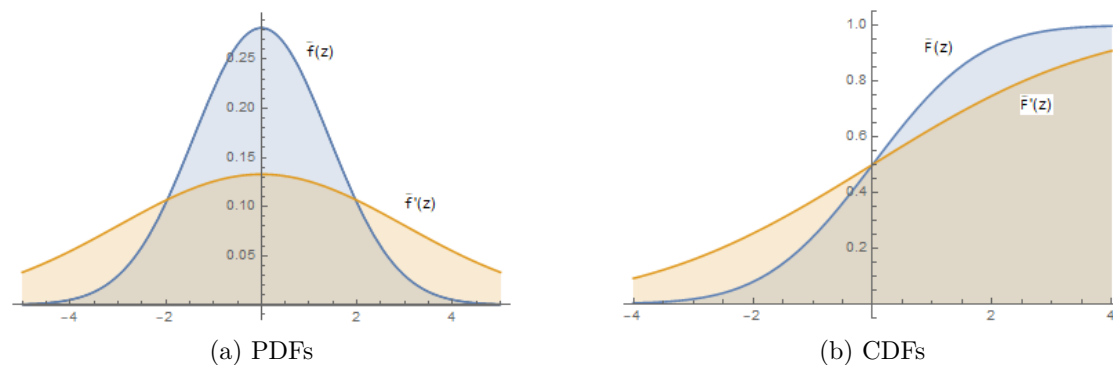


Figure 3.1: Mean-preserving spread of Normal distribution

⁴For example, if $v \sim \mathcal{N}(\mu_v, \sigma_v^2)$ and $o \sim \mathcal{N}(\mu_o, \sigma_o^2)$, then $z \sim \mathcal{N}(\mu_v - \mu_o, \sigma_v^2 + \sigma_o^2)$. Hence, an increase in either σ_v^2 or σ_o^2 affect the distribution of z , which becomes more disperse.

Proposition 3.3.6. *Suppose that the probability density function of the signals $\tilde{f}(z)$ is continuous and symmetrically distributed around its mean of zero. Let $\tilde{f}'(z)$ be a mean preserving spread of $\tilde{f}(z)$ and $F(0) = F'(0)$, then the expected aggregate of the payments $\hat{B}'_O(n) > \hat{B}_O(n)$.*

Proof. As $\tilde{F}(0) = \tilde{F}'(0)$, we can focus on the the expected value of $\hat{B}_O(k)$ and $\hat{B}'_O(k)$, which equal the second order static of k draws from $\frac{\tilde{F}(z) - \tilde{F}(0)}{1 - \tilde{F}(0)}$ and $\frac{\tilde{F}'(z) - \tilde{F}'(0)}{1 - \tilde{F}'(0)}$, respectively. Further, because $\tilde{f}'(z)$ is the mean preserving spread of $\tilde{f}(z)$, then $\tilde{F}'(z)$ first order stochastic dominates $\tilde{F}(z)$ in $z \in (0, \bar{z})$. This implies that the expected value of the second order statics of $\tilde{F}'(z)$ is greater then the one of $\tilde{F}(z)$ for all k .

□

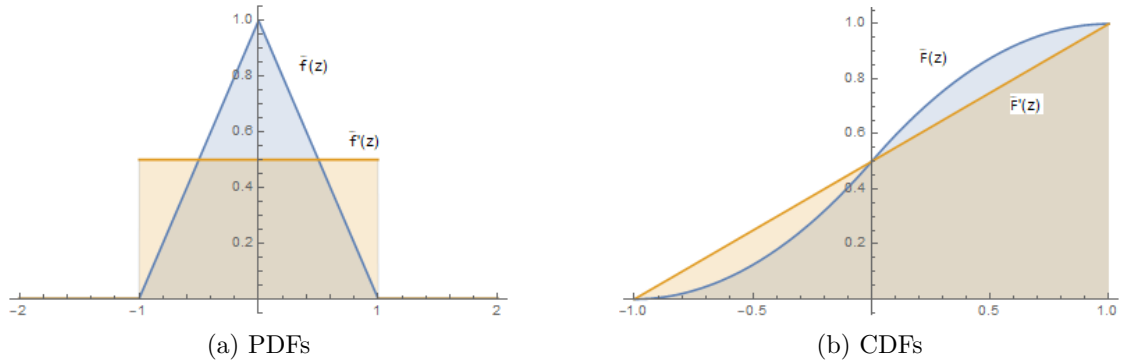


Figure 3.2: Mean-preserving spread of a Triangular Distribution

In other words, a mean preserving spread around zero does not affect the probability that a player bids zero, but it increases the probability the participants receive a higher signal of z . Overall, the expected aggregates of the payments increases. Figure (3.3) shows the expected aggregates as a function of n players of the mean preserving spread occurred in figures (3.1) and (3.2).

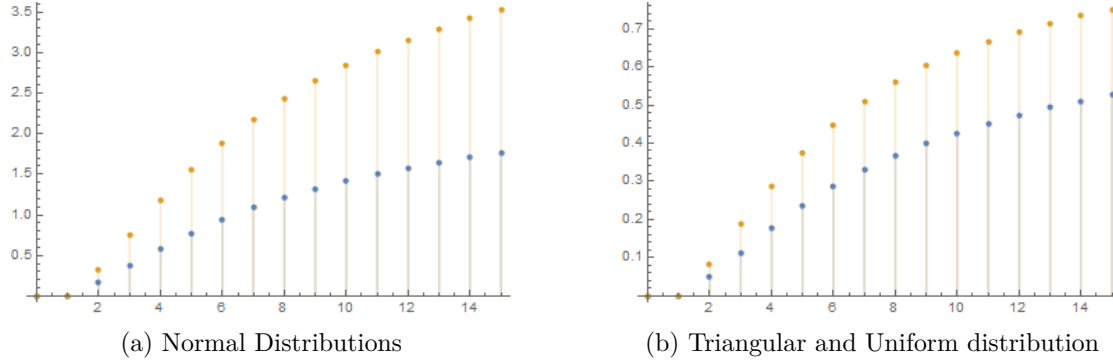


Figure 3.3: Expected aggregate of the payments as a function of n

3.4 Conclusion

In this paper, we investigate a new type of war of attrition in which players receive two signals: the valuation of the prize and the valuation of the outside option. We show that, if players' valuation of the prize is always greater than the valuation of the outside option, our model can be reduced to a standard model of war of attrition. If we allow for the possibility that a player may prefer the outside option over the prize, then there are two cases to be considered. First, only a subset of players compete, those with a preference for the prize. We call "participants" this set of players. Second, all players participate in the auction, and those with a preference for the outside option bid zero. As we show, this subtle difference creates a considerable divergence in players' equilibrium strategies.

In the first scenario, the number of bidders is public knowledge. So, participants know the exact number of bids they have to beat. We describe the equilibrium strategy and show that the expected aggregate of the payments of the auction is the weighted average of the expected aggregate of the payments for any number of participants. The number of participants follows a binomial distribution, where the

probability of success corresponds to the probability that a player prefers the prize over the outside option. Assuming that players know the number of bidders they face, we undertake comparative statics for a specific case of distribution functions. Our analysis shows that a more spread out distribution of the signals can increase the expected aggregate of the payments.

The second scenario is equivalent to a game with stochastic participation. A player interested in winning the prize does not know how many players will bid a strictly positive amount. On the contrary, what she does know is the probability that a random player makes a positive bid, as it corresponds to the probability that a player's valuation of the prize is higher than the valuation of the outside option. Using this information, a player derives the probability for every number of strictly positive bids she could compete against. The probability of every number of possible participants follows a binomial distribution since draws are independent. Finally, players choose one bidding strategy for all possible events. Such equilibrium strategy is a weighted average of the strategies they would adopt knowing the number of strictly positive bids.

An interesting open question is the comparison of the expected aggregate generated by the two situations. For example, whether a regulator who wants to minimise total expenditures should disclose or reveal the number of companies interested to get the monopoly of a market. We leave it to future research.

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